63-4-5

5043

AFOSR

414127

CATALOGED BY DDC
AS AD No. 414127

NORTH CAROLINA STATE COLLEGE

DEPARTMENT OF MATHEMATICS

APPLIED MATHEMATICS RESEARCH GROUP **RALEIGH**

AUG 24 19..

This research was supported by the Applied Mathematics Division, AFOSR, SRMA under Contract/Grant AF 49(638)-1159

Dual Series Relations

by

Ian N. Sneddon

Contract No. AF 49(638)-1159
File No. PSR-13/1

March, 1963

SCHOOL OF PHYSICAL SCIENCES AND APPLIED MATHEMATICS

DUAL SERIES RELATIONS

Lectures given in the Department of Mathematics, North Carolina State College, Raleigh in March 1963

by

IAN N. SNEDDON

Simson Professor of Mathematics in The University of Glasgow

PREFACE

This project report is based on five lectures given by Dr. I. N. Sneddon at North Carolina State College in March, 1963. The results of the research reported here have been successfully applied to the solution of certain crack problems in the mathematical theory of elasticity; some of these applications will appear in a subsequent project report and also are expected to appear in appropriate journals.

Copies of this report are being distributed as directed; this project is sponsored by AFOSR, ARO, and ONR through the Joint Services Advisory Group. The contract is currently AF 49(638)-1159.

John W. Cell
Project Director

Contents

DUAL SERIES RELATIONS

1. Introduction.

The present report is based on a series of five lectures given in the Mathematics Department of North Carolina State College in March 1963. The aim of theses lectures was to present a connected account of some recent researches on dual series relations, in particular of some work done in the University of Glasgow by the author and Dr. R. P. Srivastav.

The name dual integral equations seems to have originated with E. C. Titchmarsh who applied it to a pair of equations of the type

$$\int_0^\infty \psi(\xi)\ \omega(\xi)\ K(x, \xi)\ d\xi = f(x), \qquad 0 \leq x < 1 \tag{1.1}$$

$$\int_0^\infty \psi(\xi)\ K(x, \xi)\ d\xi = g(x), \qquad x \quad 1 \tag{1.2}$$

in which the weight function $\omega(\xi)$, the kernel $K(\xi,x)$ and the free terms $f(x)$, $g(x)$ are all prescribed and the object is to determine the unknown function $\psi(\xi)$. A survey of methods of solution of equations of this type was given recently (Sneddon, 1962).

In this report we are concerned with the series analogue of these equations, i.e. with the problem of determining a sequence of constants $\{a_n\}$ satisfying the dual relations

$$\sum_{n=1}^{\infty} a_n\ \omega(\lambda_n)\ K(x, \lambda_n) = f(x), \qquad 0 \leq x < c \tag{1.3}$$

$$\sum_{n=1}^{\infty} a_n\ K(x, \lambda_n) = g(x), \qquad c < x \leq 1 \tag{1.4}$$

where the weight function $\omega(\lambda_n)$, the kernel $K(x, \lambda_n)$ and the free terms $f(x)$, $g(x)$ are all prescribed and $\{\lambda_n\}$ is the sequence of positive zeros of a given transcendental function $j(\lambda)$. Relations of this type seem to have been discussed first by Cooke and Tranter (1959) who discussed the case in which $\omega(\lambda_n) = \lambda_n^{\ p}$, $K(x, \lambda_n) = J_v(\lambda_n x)$, $j(\lambda) = J_v(\lambda)$.

The two types of dual relation can be described within one framework. Suppose that the problem is that of determining a function ψ defined on a set I and that

$$L_1\ (x,\xi)\ \psi(\xi) = f_1(x), \qquad x \in J_1 \tag{1.5}$$

$$L_2\ (x,\xi)\ \psi(\xi) = f_2(x), \qquad x \in J_2 \tag{1.6}$$

where the linear operators L_1 and L_2 are defined on $J \times I$ where $J = J_1 \cup J_2$. An analysis of this general problem has recently been given by W. E. Williams (1963) but we shall not discuss it here. Our concern is with the solution of dual relations of the type (1.3), (1.4) for special values of the weight function and the kernel.

The work described in this report arose out of the analysis of certain physical problems so we begin in (§ 2) by reviewing some of the situations in which dual series relations occur in the analysis of mixed boundary value problems in mathematical physics. The solution of these dual series relations involves complicated mathematical analysis. With a view to shortening the proofs by separating out some frequently occurring pieces of formal calculation we separate out (in §3) some of the basic mathematical techniques. In §§ 4, 5 we consider the various kinds of dual series relations in which the kernel $K(x, \lambda_n)$ is a Bessel function of the first kind. The relations are essentially of two types; in the first (§4) the function $j(\lambda)$ is the Bessel function $J_\nu(\lambda)$ and the series involved are therefore Fourier-Bessel series, while in the second (§ 5) the function $j(\lambda)$ is the linear combination $\lambda J_\nu'(\lambda) + HJ_\nu(\lambda)$ and the series involved are Dini series. In §§6,7 we discuss dual relations involving trigonometrical series which are of use in the solution of certain two-dimensional boundary value problems. In the next section (§ 8) we discuss a class of dual series relations involving series of associated Legendre functions, using the method of Collins (1961). Series relations of this kind which arise in the analysis of problems relating to spherical caps form a particular case of the problem of solving certain dual series relations involving series of Jacobi polynomials. We conclude our treatment by outlining (in §8) Srivastav's method of solving equations of this type.

2. The Occurrence of Dual Series Relations in Mathematical Physics.

We begin by considering some of the circumstances in which dual series relations arise in the analysis of boundary value problems in mathematical physics.

2.1. Electrified Disk in a Grounded Cylinder.

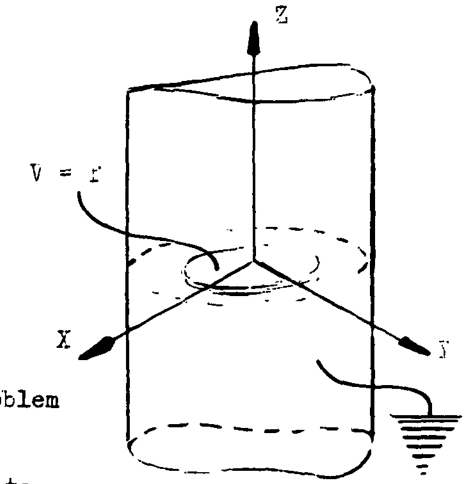

Fig. 1

Suppose that an electrified disk is situated in a plane normal to the axis of an infinite circular cylinder its centre lying on that axis. The cylinder is assumed to be grounded, and the potential of the disk is prescribed. We take the origin of coordinates at the centre of the disk and the z-axis along the axis of the cylinder. For convenience we can take the unit of length in our problem to be the radius of the disk; we may then take the radius of the cylinder to be $a > 1$. The problem is to find the potential V of the electrostatic field in the interior of the cylinder. If we introduce cylindrical coordinates (ρ, ϕ, z) then $V(\rho, \phi, z)$ must be a solution of Laplace's equation

$$\frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2.1}$$

where $V(\rho, \phi, z) \longrightarrow 0$ as $|z| \longrightarrow \infty$ and

$$V(\rho, \phi, 0) = f(\rho, \phi), \quad 0 \leqslant \rho < 1,$$

and

$$V(a, \phi, z) = 0, \qquad |z| > 0. \tag{2.2}$$

Using the superposition principle we see that if we can solve the boundary value in which

$$V(\rho, \phi, 0) = f_\nu(\rho)\cos(\nu\phi + \alpha_\nu), \quad 0 \leqslant \rho < 1, \tag{2.3}$$

we can solve the boundary value problem in which $V(\rho, \phi, 0) = f(\rho, \phi)$

Instead of solving the Dirichlet problem for the space between the disk and the circular cylinder we can reduce the problem to a mixed boundary value problem for the semi-infinite cylinder $z \geqslant 0$, $0 \leqslant \rho \leqslant a$ when the boundary conditions (2.2), (2.3) are supplemented by the condition

$$\frac{\partial V}{\partial z} = 0, \quad z = 0, \quad 1 < \rho < a. \tag{2.4}$$

The function

$$V(\rho, \phi, z) = \cos(\nu\phi + \alpha_\nu)\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_\nu(\lambda_n \rho) e^{-\lambda_n z} \tag{2.5}$$

satisfies the harmonic equation (2.1); it satisfies the condition $V \to 0$ as $z \to \infty$ and the boundary condition (2.2) provided that $\lambda_1, \lambda_2, \lambda_3, \ldots$ are chosen to be the positive roots of the transcendental equation

$$J_\nu(\lambda a) = 0. \tag{2.6}$$

To satisfy the boundary conditions (2.3), (2.4) we must choose the constants a_1, a_2, a_3, ... to satisfy the relations

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_\nu(\lambda_n \rho) = f_\nu(\rho), \quad 0 \leqslant \rho < 1 \tag{2.7}$$

$$\sum_{n=1}^{\infty} a_n J_\nu(\lambda_n \rho) = 0, \qquad 1 < \rho \leqslant a. \tag{2.8}$$

In particular if the prescribed potential of the disk is symmetrical about the z-axis, i.e. if $V(\rho, \phi, 0) = f(\rho)$, for $0 \leqslant \rho < 1$, we may take $\nu = 0$ in these equations to obtain the pair of equations

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\lambda_n \rho) = f(\rho), \qquad 0 \leqslant \rho < 1, \tag{2.9}$$

$$\sum_{n=1}^{\infty} a_n J_0(\lambda_n \rho) = 0, \qquad 1 < \rho \leqslant a, \tag{2.10}$$

where now the $\{\lambda_n\}$ are the positive zeros of the function $J_1(\lambda a)$.

Pairs of equations of the type (2.7), (2.8) or the type (2.9), (2.10) in which the sequence of constants $\{a_n\}$ is determined by a pair of equations, one of which is valid in one segment of the positive real axis and the other of which is valid over another segment, are called <u>dual series equations</u>.

2.2. The Reissner-Sagoci Problem for a Semi-Infinite Cylinder.

The standard Reissner-Sagoci problem is that of determining the components of stress and displacement in the interior of the semi-infinite elastic solid $z \geqslant 0$ when a circular area of the boundary surface $z = 0$ is forced to rotate through an angle α about an axis which is normal to the undeformed plane surface of the solid; it is assumed that the part of the boundary surface which lies outside this circle is free from stress. (Cf. Sneddon, 1951, p.500). Recently Sneddon and Srivastav (1963) have generalized this problem to the case in which the semi-infinite elastic body has a cylindrical boundary whose cross section is a circle of radius a.

It has been shown by Reissner (1937) that in this case the only non-vanishing component u_ϕ, where we use cylindrical coordinates (ρ, ϕ, z); all the components of the stress tensor vanish except $\sigma_{z\phi}$ and $\sigma_{\rho\phi}$ which are related to u_ϕ through the equations

$$\sigma_{z\phi} = \mu \frac{\partial u_\phi}{\partial z}, \qquad \sigma_{\rho\phi} = \mu \rho \frac{\partial}{\partial \rho}\left(\frac{u_\phi}{\rho}\right), \tag{2.11}$$

where μ denotes the rigidity modulus. To satisfy the equation of equilibrium

$$\frac{\partial \sigma_{\rho\phi}}{\partial \rho} + \frac{\partial \sigma_{z\phi}}{\partial z} + \frac{2\sigma_{\rho\phi}}{\rho} = 0$$

we must choose a form for u_ϕ satisfying the equation

$$\frac{\partial^2 u_\phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u_\phi}{\partial \rho} - \frac{u_\phi}{\rho^2} + \frac{\partial^2 u_\phi}{\partial z^2} = 0. \tag{2.12}$$

The boundary conditions of the problem are simplified if we choose the radius of the prescribed circle to which the rotation is applied to be our unit of length, On the plane boundary $z = 0$ we then have the boundary conditions

$$u_\phi = f(\rho), \quad 0 \leqslant \rho < 1; \qquad \sigma_{z\phi} = 0, \quad 1 < \rho < a, \tag{2.13}$$

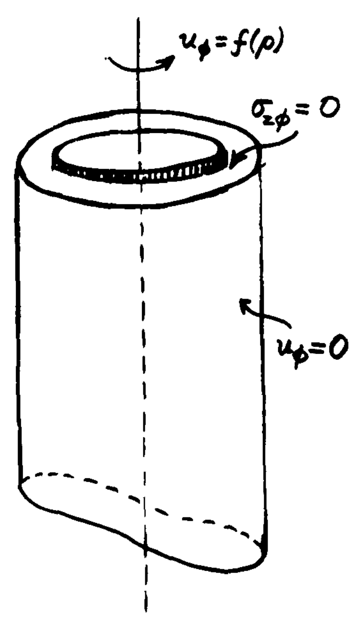

Fig. 2

where the function $f(\rho)$ is prescribed. In the case of most immediate physical interest $f(\rho) = \alpha\rho$, where α is a constant, but it is of value to derive the solution in the general case.

In the first instance we assume that the cylindrical wall of the cylinder is rigidly clamped; i.e. we assume that

$$u_\phi = 0, \quad \rho = a, \quad z > 0. \tag{2.14}$$

We also assume that u_ϕ, $\sigma_{z\phi}$, $\sigma_{\rho\phi}$ all tend to zero as $z \to \infty$. (Cf. Fig. 2).

If we observe that

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + \frac{\partial}{\partial z^2}\right)\frac{\partial \psi}{\partial \rho} = \frac{\partial}{\partial \rho}\nabla^2\psi,$$

we see that

$$u_\phi = \frac{\partial \psi}{\partial \rho} \tag{2.15}$$

is a solution of equation (2.12) provided that ψ is a harmonic function. In this case the expressions for the non-vanishing components of the stress tensor assume the forms

$$\sigma_{z\phi} = \mu \frac{\partial^2 \psi}{\partial \rho \partial z}, \qquad \sigma_{\rho\phi} = \mu \rho \frac{\partial}{\partial \rho}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial \rho} \right). \tag{2.16}$$

If we write

$$\psi(\rho, z) = -\sum_{n=1}^{\infty} \lambda_n^{-2} a_n e^{-\lambda_n z} J_0(\lambda_n \rho) \tag{2.17}$$

in equations (2.15) and (2.16), where $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are the positive zeros of $J_1(\lambda a)$, then

$$u_\phi = \sum_{n=1}^{\infty} \lambda_n^{-1} a_n e^{-\lambda_n z} J_1(\lambda_n \rho), \tag{2.18}$$

$$\sigma_{\phi z} = -\mu \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} J_1(\lambda_n \rho), \tag{2.19}$$

so that equation (2.14) is satisfied and the mixed boundary conditions (2.13) will be satisfied if we choose the constants a_n to satisfy the dual series equations

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_1(\lambda_n \rho) = f(\rho), \quad 0 \leqslant \rho < 1, \tag{2.20}$$

$$\sum_{n=1}^{\infty} a_n J_1(\lambda_n \rho) = 0, \quad 1 < \rho \leqslant a, \tag{2.21}$$

where, it will be remembered, the $\{\lambda_n\}$ are the positive zeros of $J_1(\lambda a)$.

If, instead of the boundary condition (2.14), we assume that the cylindrical boundary is stress-free, i.e. if we assume that

$$\sigma_{\rho\phi} = 0, \quad \rho = a, \quad z > 0, \tag{2.22}$$

it is necessary to consider, not a semi-infinite cylinder, but a very long one of length $\delta >> a$. We assume that the base of this cylinder has zero displacement, i.e. that

$$u_\phi = 0, \quad z = \delta, \quad 0 \leqslant \rho \leqslant a. \tag{2.23}$$

The conditions (2.13), (2.22), (2.23) can be realized by considering the distortion of a cylinder which is rigidly attached to a fixed rigid foundation $z = \delta$ and is deformed by the application of a torque to a

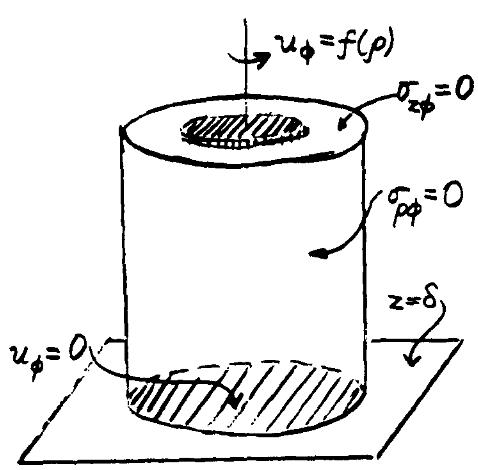

Fig. 3

circular patch of its other plane surface (e.g. through the rotation of a rigid disk attached to it) while the remainder of its surface is free from stress (Cf. Fig. 3).

If we assume the form

$$u_\phi = a_0 \rho(\delta - z) + \sum_{n=1}^{\infty} \lambda_n^{-1} a_n \frac{\sinh \lambda_n(\delta - z)}{\sinh \lambda_n \delta} J_1(\lambda_n \rho) \tag{2.24}$$

for the displacement, then we see from equations (2.11) that the stress components are given by the equations

$$\sigma_{z\phi} = -\mu a_0 \rho - \mu \sum_{n=1}^{\infty} a_n \frac{\cosh \lambda_n(\delta - z)}{\sinh \lambda_n \delta} J_1(\lambda_n \rho), \tag{2.25}$$

$$\sigma_{\rho\phi} = -\sum_{n=1}^{\infty} a_n \frac{\sinh \lambda_n(\delta - z)}{\sinh \lambda_n \delta} J_2(\lambda_n \rho) \tag{2.26}$$

It follows immediately from these equations that the boundary condition (2.23) is satisfied and that the boundary condition (2.22) will be satisfied provided the λ_n's are the positive zeros of $J_2(\lambda a)$. Also, since on $z = 0$,

$$u_\phi = a \rho \delta + \sum_{n=1}^{\infty} a_n^{-1} J_1(\lambda_n \rho)$$

$$\sigma_{z\phi} = -\mu a_0 \rho - \mu \sum_{n=1}^{\infty} a_n \coth(\lambda_n \delta) J_1(\lambda_n \rho)$$

it follows that the boundary conditions (2.13) will be satisfied if the coefficients a_n satisfy the dual series relations

$$a_0 \rho \delta + \sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_1(\lambda_n \rho) = f(\rho), \quad 0 \leqslant \rho < 1 \tag{2.27}$$

$$a_0 \rho + \sum_{n=1}^{\infty} a_n \coth(\lambda_n \delta) J_1(\lambda_n \rho) = 0, \quad 1 < \rho \leqslant a.$$

The last pair of equations is the pair appropriate to the deformation of a cylinder of finite length δ. If we are considering a very long cylinder ($\delta \gg a$) then we may replace the factors $\coth(\lambda_n \delta)$ occurring in the series on the left-hand side of equation (2.28) by unity. For a long cylinder the problem is therefore reduced to the solution of the dual series relations

$$a_0\rho\delta + \sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_1(\lambda_n\rho) = f(\rho), \quad 0 \leq \rho < 1, \tag{2.29}$$

$$a_0 + \sum_{n=1}^{\infty} a_n J_1(\lambda_n\rho) = 0, \qquad 1 < \rho \leq a, \tag{2.30}$$

where δ is a known constant and the λ_n's are the positive zeros of $J_2(\lambda a)$.

2.3. Problems in the Conduction of Heat.

Similar equations arise in problems concerned with the conduction of heat in cylinders. If $\theta(\rho, \phi, z)$ is the deviation of the temperature at a point with cylindrical coordinates (ρ, ϕ, z) from a standard temperature θ_0 then it is well-known that in the steady state θ must be a harmonic function in the region considered, so that a solution appropriate to the semi-infinite cylinder $0 \leq \rho \leq a$, $z \geq 0$, in which there is an axisymmetric flow heat is

$$\theta(\rho, z) = \sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\rho\lambda_n) e^{-\lambda_n z} \tag{2.31}$$

From this equation it follows that the rate of flow of heat per unit area across the plane $z = 0$ out of the solid is

$$\left(+ k \frac{\partial\theta}{\partial z}\right)_{z=0} = -k \sum_{n=1}^{\infty} a_n J_0(\rho\lambda_n), \tag{2.32}$$

where k is the conductivity, while that across the cylindrical surface $\rho = a$ out of the solid is

$$\left(- k \frac{\partial\theta}{\partial\rho}\right)_{\rho=a} = k \sum_{n=1}^{\infty} a_n e^{-\lambda_n z} J_1(a\lambda_n) \tag{2.33}$$

The constants λ_n are, as yet, unspecified. Their values will be determined by the boundary conditions on the surface $\rho = a$. There are three important cases to be considered.

(i) In the first case we assume that the temperature of the surface $\rho = a$ is maintained at the constant value θ_0 so that $\theta(a, z) = 0$. From equation (2.31) it follows that in this case the λ_n's are chosen to be the positive zeros of $J_0(\lambda a)$.

If the plane surface $z = 0$ is heated in such a way that the temperature is prescribed over the circle $0 \leq \rho < 1$ and the remainder of the surface is insulated then we obtain the pair of dual integral equations

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\rho\lambda_n) = f(\rho), \quad 0 \leqslant \rho < 1 \tag{2.34}$$

$$\sum_{n=1}^{\infty} a_n J_0(\rho\lambda_n) = 0, \quad 1 < \rho \leqslant a, \tag{2.35}$$

where the function $f(\rho)$ is prescribed.

On the other hand if the surface is heated in such a way that there is a prescribed flux of heat into the circle $0 \leqslant \rho < 1$ and the remainder of the surface is maintained at the standard temperature θ_0, then the a_n must satisfy the equations

$$\sum_{n=1}^{\infty} a_n J_0(\rho\lambda_n) = g(\rho), \quad 0 \leqslant \rho < 1 \tag{2.36}$$

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\rho\lambda_n) = 0, \quad 1 < \rho \leqslant a, \tag{2.37}$$

where the function $g(\rho)$ is prescribed.

Equations of a more complicated type arise if we have radiation from the surface $z = 0$ into a medium maintained at a constant temperature θ_0.

By Newton's law of cooling we then have that on $z = 0$

$$\left(k\frac{\partial \theta}{\partial z}\right)_{z=0} = H\theta(\rho, 0)$$

where H is a constant. From equations (2.31) and (2.32) we then find that

$$\left(H\theta - k\frac{\partial\theta}{\partial z}\right)_{z=0} = k\sum_{n=0}^{\infty} a_n(1 + h\lambda_n^{-1})J_0(\rho\lambda_n)$$

where $h = H/k$.

Hence if the temperature is prescribed over the circle $0 \leqslant \rho < 1$ of the surface $z = 0$ and over the rest of the surface there is radiation into a medium maintained at temperature θ_0, the constants a_n occurring in the solution (2.31) must satisfy the dual series equations

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\lambda_n\rho) = f(\rho), \quad 0 \leqslant \rho < 1, \tag{2.38}$$

$$\sum_{n=1}^{\infty} a_n(1 + h\lambda_n^{-1})J_0(\lambda_n \rho) = 0, \quad 1 < \rho \leqslant a. \tag{2.39}$$

On the other hand if the heat flux is prescribed over the circle $z = 0$, $0 \leqslant \rho < 1$ and we have the radiation condition over the zone $1 < \rho \leqslant a$, we see that the equations determining the coefficients a_n are

$$\sum_{n=1}^{\infty} a_n J_0(\lambda_n \rho) = g(\rho), \quad 0 \leqslant \rho < 1, \tag{2.40}$$

$$\sum_{n=1}^{\infty} (1 + h\lambda_n^{-1})\, a_n J_0(\lambda_n \rho) = 0, \quad 1 < \rho \leqslant a. \tag{2.41}$$

(ii) In the second case we assume that the cylindrical surface $\rho = a$ is insulated against the flow of heat. From equation (2.33) it follows that the λ_n's in this case are the positive zeros of $J_1(\lambda a)$. We again get boundary value problems of the types considered in (i). The dual series relations are exactly the same as those listed above except that the λ_n are now the positive zeros of $J_1(\lambda a)$.

(iii) Finally if there is radiation from the cylindrical surface into a medium maintained at the fixed temperature θ_0 we have, as a consequence of Newton's law of cooling, the equation

$$\left(H\theta + k\frac{\partial \theta}{\partial \rho}\right)_{\rho = a} = 0. \tag{2.42}$$

Substituting from equations (2.31) and (2.33) we find that

$$\left(H\theta + k\frac{\partial \theta}{\partial \rho}\right)_{\rho = a} = k\sum_{n=1}^{\infty} \lambda_n^{-1} a_n\left[hJ_0(\lambda_n a) + \lambda_n J_0'(\lambda_n a)\right] e^{-\lambda_n z}$$

from which it follows that the conditions (2.42) is satisfied if we choose the λ_n occurring in the solution (2.31) to be the positive roots of the transcendental equation

$$hJ_0(\lambda a) + \lambda J_0'(\lambda a) = 0. \tag{2.43}$$

Dual series involving trigonometric series arise in the solution of problems concerning the conduction of heat in the long strip $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \delta$. If we suppose that the temperature deviation $\theta(x, y)$ is prescribed over the segment $0 \leqslant x < c$, $y = 0$ and that the remainder of the boundary of the strip is insulated against the flow of heat across it then it is readily shown that the solution of the problem will be

$$\theta(x, y) = \tfrac{1}{2}a_0(\delta - y) + \sum_{n=1}^{\infty} n^{-1} a_n \cos(nx) \frac{\cosh n(\delta - y)}{\cosh n\delta}, \; (0 \leqslant x \leqslant \pi, 0 \leqslant y \leqslant \delta), \tag{2.44}$$

provided that we can determine a sequence of constants $\left\{a_n\right\}$ to satisfy the dual series relations

$$\tfrac{1}{2} a_0 \delta + \sum_{n=1}^{\infty} n^{-1} a_n \cos(nx) = f(x), \qquad 0 \leqslant x < c, \tag{2.45}$$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \tanh(n\delta) \cos(nx) = 0, \qquad c < x \leqslant \pi, \tag{2.46}$$

where the function $f(x)$ is prescribed. If $\delta >> \pi$ we can replace $\tanh(n\delta)$ on the left-hand side of equation by unity to obtain the dual series relations

$$\tfrac{1}{2} a_0 \delta + \sum_{n=1}^{\infty} n^{-1} a_n \cos(nx) = f(x), \qquad 0 \leqslant x < c, \tag{2.47}$$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = 0, \qquad c < x \leqslant \pi. \tag{2.48}$$

On the other hand if the conditions on the sides $x = 0$, $x = \pi$ are replaced by the conditions $\theta(0, y) = \theta(\pi, y) = 0$ the solution appropriate to the semi-infinite strip $0 \leqslant x \leqslant \pi$, $y \geqslant 0$ is

$$\theta(x, y) = \sum_{n=1}^{\infty} n^{-1} a_n \sin(nx) e^{-ny}, \tag{2.49}$$

where the constants $\left\{a_n\right\}$ satisfy the dual series equations

$$\sum_{n=1}^{\infty} n^{-1} a_n \sin(nx) = f(x), \qquad 0 \leqslant x < c, \tag{2.50}$$

$$\sum_{n=1}^{\infty} a_n \sin(nx) = 0, \qquad c < x \leqslant \pi. \tag{2.51}$$

2.4. Some Boundary Value Problems for a Spherical Cap.

Dual series relations involving Fourier-Legendre series arise in the analysis of some electrostatic and hydrodynamic boundary value problems for a spherical cap.

The first problem we consider is that of determining the electrostatic potential u due to a thin spherical cap maintained at a prescribed potential. If we use spherical polar coordinates (r, θ, ϕ) referred to the centre of the sphere as origin and the axis of symmetry of the cap as polar axis (Oz in Fig. 4), we can describe the cap by the equations $r = a$, $0 \leqslant \theta \leqslant \alpha$. On the assumption that the prescribed potential is symmetrical about the axis of the cap, i.e. that $u = f(\theta)$ on the spherical cap, we may take u to be an axisymmetric solution of Laplace's equation

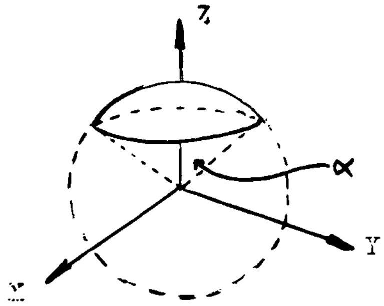

Fig. 4

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0$$

satisfying the conditions

(i) $u = f(\theta)$ on $r = a$, $0 \leqslant \theta \leqslant \alpha$; and u is continuous across $r = a$;

(ii) $\partial u/\partial r$ is continuous across the spherical region $r = a$, $\alpha < \theta < \pi$;

(iii) $u = O(r^{-1})$ for large r.

A solution of Laplace's equation satisfying the continuity condition on u across $r = a$ and the condition (iii) is

$$u(r, \theta) = \begin{cases} \sum_{n=0}^{\infty} a_n \left(\frac{r}{a}\right)^n P_n(\cos\theta), & 0 \leqslant r \leqslant a \\ \sum_{n=0}^{\infty} a_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta), & r \leqslant a \end{cases}$$

where to satisfy the conditions (i) and (ii) the constants a_n must be chosen to be solutions of the dual series relations

$$\sum_{n=0}^{\infty} a_n P_n(\cos\theta) = f(\theta), \qquad (0 \leqslant \theta < \alpha), \tag{2.52}$$

$$\sum_{n=0}^{\infty} (2n+1)\, a_n P_n(\cos\theta) = 0, \qquad (\alpha < \theta \leqslant 2\pi). \tag{2.53}$$

A similar boundary value problem involves the determination of the velocity potential Φ due to the motion of a spherical cap along the direction of its axis with constant velocity U in a perfect fluid at rest at large distances from the cap. If we use the above coordinate system then $\Phi = \Phi(r, \theta)$ must be an axisymmetrical solution of Laplace's equation such that, for large values of r, $\Phi = O(r^{-2})$ and $r\partial\Phi/\partial r$ is continuous across the sphere $r = a$; further it must satisfy the condition

$$r \frac{\partial\Phi}{\partial r} = - U a \cos\theta, \quad r = a, \quad 0 \leqslant \theta \leqslant \alpha \tag{2.54}$$

while on the spherical region $r = a$, $\alpha \leqslant \theta \leqslant \pi$, Φ must be continuous (Lamb, 1953, p.160). An axisymmetric solution which is $O(r^{-2})$ for large r and has $r\partial\Phi/\partial r$ continuous across $r = a$ is given by the equations

$$\Phi(r,\theta) = \begin{cases} \displaystyle\sum_{n=1}^{\infty}(n+1)a_n\left(\frac{r}{a}\right)^n P_n(\cos\theta), & (r < a), \\ \displaystyle -\sum_{n=1}^{\infty} n\, a_n\left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta), & (r > a). \end{cases}$$

The condition (2.54) and the continuity of Φ across $r = a$, $\alpha \leqslant \theta \leqslant \pi$ are satisfied if the constants a_n satisfy the dual series equations

$$\sum_{n=1}^{\infty} n(n+1)a_n P_n(\cos\theta) = - Ua\cos\theta, \; 0 \leqslant \theta < \alpha, \tag{2.55}$$

$$\sum_{n=1}^{\infty} (2n+1)a_n P_n(\cos\theta) = 0, \qquad \alpha < \theta \leqslant \pi. \tag{2.56}$$

On the other hand if we suppose that the cap moves through the fluid in the direction $\phi = 0$ perpendicular to its axis of symmetry with uniform velocity U. The velocity potential $\Phi(r, \theta, \phi)$ for the motion must satisfy Laplace's equation

$$r^2\frac{\partial^2\Phi}{\partial r^2} + 2r\frac{\partial\Phi}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

be such that $\Phi = O(r^{-2})$ for large r, Φ and $\partial\Phi/\partial r$ are continuous over the surface $r = a$, $\alpha < \theta \leqslant 2\pi$ while

$$r\frac{\partial\Phi}{\partial r} = - U a \sin\theta\cos\phi\,. \tag{2.57}$$

It is easily shown that

$$\Phi(r,\theta,\phi) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}(n+2)a_n\left(\frac{r}{a}\right)^{n+1} T_{n+1}^{-1}(\cos\theta)\cos\phi, & 0 \leq r < a \\ \displaystyle -\sum_{n=0}^{\infty}(n+1)a_n\left(\frac{a}{r}\right)^{n+2} T_{n+1}^{-1}(\cos\theta)\cos\phi, & r > a \end{cases}$$

where the coefficients a_n satisfy the dual series relations

$$\sum_{n=1}^{\infty}(n+1)(n+2)a_n T_{n+1}^{-1}(\cos\theta) = -U a \sin\theta, \quad 0 \leq \theta < \alpha$$

$$\sum_{n=1}^{\infty}(2n+3)a_n T_{n+1}^{-1}(\cos\theta) = 0, \qquad \alpha < \theta < \pi .$$

3. Mathematical Preliminaries.

The solution of the various types of dual series relations which we have derived in §2 is not a simple matter. The analysis is complicated even in the simplest case and it is preferable to shorten the proofs by separating out various bits of formal analysis. This is what we shall do in this section. We begin by listing some properties of operators of fractional integration.

3.1. The Erdelyi-Kober Operators.

We shall make use of the Erdelyi-Kober operators $I_{\eta,\alpha}$ as modified by Sneddon (1962). If η and α are real, $\alpha > 0$ and $\eta > -\frac{1}{2}$ we define the operator $I_{\eta,\alpha}$ by the equation

$$I_{\eta,\alpha}\left\{f(u);\; x\right\} = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x u^{2\eta+1}(x^2-u^2)^{\alpha-1}f(u)\,du \qquad (3.1)$$

while if $\alpha < 0$ and n is the least positive integer such that $\alpha + n > 0$ we define it by the equation

$$I_{\eta,\alpha}\left\{f(u);\, x\right\} = x^{-2\eta-2\alpha-1}\,\mathcal{D}_x^{\,n}\, x^{2n+2\alpha+2\eta+1} I_{\eta,\alpha+n}\left\{f(u),\, x\right\}, \qquad (3.2)$$

where $\mathcal{D}_x$ denotes the operator defined by the equation

$$\mathfrak{D}_x f = \frac{1}{2}\frac{d}{dx}\left(\frac{f}{x}\right). \tag{3.3}$$

We shall in fact only encounter the case in which $-1 < \alpha < 0$ when equation (3.2) reduces to the simple form

$$I_{\eta,\alpha}\left\{f(u);\ x\right\} = \frac{x^{-2\eta-2\alpha-1}}{\Gamma(1+\alpha)}\frac{d}{dx}\int_0^x (x^2-u^2)^{\alpha}u^{2\eta+1} f(u)du \tag{3.4}$$

A much used property of these operators is that

$$I_{\eta,\alpha}\left\{cu^{2\beta} f(u);\ x\right\} = cx^{2\beta} I_{\eta+\beta,\alpha}\left\{f(u);\ x\right\} \tag{3.5}$$

We shall also require the fact that $I_{\eta,\alpha}$ has an inverse defined by the equation

$$I^{-1}_{\eta,\alpha} = I_{\eta+\alpha,-\alpha}, \tag{3.6}$$

and the property

$$I_{\eta,\alpha} I_{\eta+\alpha,\beta} = I_{\eta,\alpha+\beta}. \tag{3.7}$$

We also note that $I_{\eta,0}$ is the identity operator.

When the function to which the operator $I_{\eta,\alpha}$ is applied is a function of more than one variable we shall modify the notation of (3.1). For example we shall use $I_{\eta,\alpha}\left\{f(u,t);\ u\rightarrow x\right\}$ to mean

$$\frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x u^{2\eta+1}(x^2-u^2)^{\alpha-1} f(u,t)du.$$

It considerably simplifies the writing down of our results if we define a function $K_{\nu,\alpha,\beta,\gamma}(\rho,t;a)$ to be the integral

$$\int_0^\infty \frac{K_\nu(ay)}{I_\nu(ay)} I_\alpha(\rho y) I_\beta(ty) y^{1+\gamma} dy$$

(when the integral converges). When it is clear from the context what ν and $\underline{a}$ are we shall write $K_{\alpha,\beta,\gamma}(\rho,t)$ for $K_{\nu,\alpha,\beta,\gamma}(\rho,t;a)$. The integral will satisfy the correct convergence conditions 'at infinity' if $2a > \rho + t$; since, in our applications $\rho < a$, $t < a$ this condition is automatically satisfied. To ensure convergence at the lower limit of integration we must have $-2\nu+\alpha+\beta+1+\gamma > -1$, i.e. we must have $\nu < 1+\frac{1}{2}(\alpha+\beta+\gamma)$.

The following relations satisfied by $K_{\alpha,\beta,\gamma}(\rho,t)$ can easily be deduced from the recurrence relations satisfied by modified Bessel functions of the first kind

(Watson, (1944), p.79) and integrals given by formulae (6), (7) on p.365 of Vol.II of Erdelyi (1954):

$$\frac{\partial}{\partial \rho}\left\{\rho^{\alpha} K_{\alpha,\beta,\gamma}(\rho, t)\right\} = \rho^{\alpha} K_{\alpha-1,\beta,\gamma+1}(\rho, t), \tag{3.8}$$

$$\frac{\partial}{\partial t}\left\{t^{\beta} K_{\alpha,\beta,\gamma}(\rho, t)\right\} = t^{\beta} K_{\alpha,\beta-1,\gamma+1}(\rho, t), \tag{3.9}$$

$$I_{\frac{1}{2}\alpha,\mu}\left\{K_{\alpha,\beta,\gamma}(\rho, t);\ \rho \to x\right\} = 2^{\mu} x^{-\mu} K_{\alpha+\mu,\beta,\gamma-\mu}(x, t), \tag{3.10}$$

$$I_{\frac{1}{2}\beta,\mu}\left\{K_{\alpha,\beta,\gamma}(\rho, t);\ t \to x\right\} = 2^{\mu} x^{-\mu} K_{\alpha,\beta+\mu,\gamma-\mu}(x, t). \tag{3.11}$$

We shall also have occasion to consider the function defined by

$$K^{*}_{\nu,H,\beta,\gamma,\delta}(u, v) = \int_0^{\infty} \frac{yK'_{\nu}(y) + HK_{\nu}(y)}{yI'_{\nu}(y) + HI_{\nu}(y)} I_{\beta}(uy)\, I_{\gamma}(vy) y^{1+\delta}\, dy \tag{3.12}$$

$(0 < u < 1,\ 0 < v < 1)$ whenever the integral converges. Using the recurrence relations satisfied by modified Bessel functions of the first kind and the results in Erdelyi (1954) quoted above we find that

$$\frac{\partial}{\partial u}\left\{u^{\beta} K^{*}_{\nu,H,\beta,\gamma,\delta}(u, v)\right\} = u^{\beta} K^{*}_{\nu,H,\beta-1,\gamma,\delta+1}(u, v), \tag{3.13}$$

$$\frac{\partial}{\partial v}\left\{v^{\gamma} K^{*}_{\nu,H,\beta,\gamma,\delta}(u, v)\right\} = v^{\gamma} K^{*}_{\nu,H,\beta,\gamma-1,\delta+1}(u, v), \tag{3.14}$$

$$I_{\frac{1}{2}\beta,\alpha}\left\{K^{*}_{\nu,H,\beta,\gamma,\delta}(u, v);\ u \to x\right\} = 2^{\alpha} x^{-\alpha} K^{*}_{\nu,H,\alpha+\beta,\gamma,\delta-\alpha}(x, v) \tag{3.15}$$

$$I_{\frac{1}{2}\gamma,\alpha}\left\{K^{*}_{\nu,H,\beta,\gamma,\delta}(u, v);\ v \to y\right\} = 2\ y^{-\alpha} K^{*}_{\nu,H,\beta,\gamma+\alpha,\delta-\alpha}(u, y) \tag{3.16}$$

3.2. Infinite Series involving Bessel Functions.

In the subsequent analysis we shall encounter functions of the type $S_{\nu,\alpha,\beta,\gamma}(\rho, t; a)$ defined as the sum of the infinite series

$$S_{\nu,\alpha,\beta,\gamma}(\rho, t; a) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{J_{\alpha}(\rho\lambda_n)\, J_{\beta}(t\lambda_n)}{J^2_{\nu+1}(a\lambda_n)} \lambda_n^{\gamma}, \tag{3.17}$$

whenever the infinite series converges. The infinite sequence $\{\lambda_n\}$ is formed of the positive zeros of the function $J_{\nu+1}(a\lambda)$. In equation (3.17), ρ and t are real variables satisfying the inequalities $0 \leq \rho, t \leq 1 < a$ and $-(\alpha+\beta+2) < \gamma \leq 0$.

Series of this type are considered in Sneddon and Srivastav (1963) but special cases were considered earlier by Tranter (1959). Tranter defines two functions by the series

$$s_1(\nu, m, k, \rho) = \sum_{s=1}^{\infty} \frac{J_{\nu+2m+k}(\lambda_s) J_\nu(\alpha_s \rho)}{\lambda_s^k J_{\nu+1}^2(a\lambda_s)}, \qquad \nu > -1 \tag{3.18}$$

$$s_2(\nu, m, n, k) = \sum_{s=1}^{\infty} \frac{J_{\nu+2m+k}(\lambda_s) J_{\nu+2n+k}(\lambda_s)}{\lambda_s^2 J_{\nu+1}^2(a\lambda_s)}, \qquad \nu -m-n-k \tag{3.19}$$

where m and n are zero or positive integers, $k > 0$, and $a > -1$. It is easily seen that, in the notation of equation (3.17),

$$s_1(\nu, m, k, \rho) = \tfrac{1}{2} a^2 S_{\nu, \nu, \nu+2m+k, -k}(\rho, 1; a), \tag{3.18a}$$

and

$$s_2(\nu, m, n, k) = \tfrac{1}{2} a^2 S_{\nu, \nu+2m+k, \nu+2n+k, -2}(1, 1; a). \tag{3.19a}$$

We shall also find that we have to deal with series of the type

$$S^*_{\nu, H, \beta, \gamma, \delta}(u, v) = 2 \sum_{m=1}^{\infty} \frac{J_\beta(u\lambda_m) J_\gamma(v\lambda_m) \lambda_m^{2+\delta}}{(\lambda_m^2 - \nu^2 + H^2) J_\nu^2(\lambda_m)} \tag{3.20}$$

$0 < u < 1$, $0 < v < 1$, where, now, $\{\lambda_m\}$ is the infinite sequence formed by the positive roots of the transcendental equation

$$\lambda J_\nu'(\lambda) + H J_\nu(\lambda) = 0, \tag{3.21}$$

H and ν being real constants, $\nu \geq -\frac{1}{2}$. Series of this type have been discussed by Srivastav (1963a).

To obtain an integral representation of the series on the right-hand side of equation (3.17) we consider the contour integral

$$\int_C F(z)\,dz$$

where the function $F(z)$ is defined by the equation

$$F(z) = \frac{J_\nu(a\,z) + i\,Y_\nu(a\,z)}{J_\nu(a\,z)} J_\alpha(\rho z)\,J_\beta(t\,z) \tag{3.22}$$

and the contour C consists of:-

(i) the portions of the positive real axis joining the points

$\delta, \lambda_1 - \delta_1; \lambda_s + \delta_s, \lambda_{s+1} - \delta_{s+1}$

$(s = 1, 2, \ldots, p-1)$; $\lambda_p + \delta_p$, R; where the δ's are small and p and R are large and such that $\lambda_p < R < \lambda_{p+1}$;

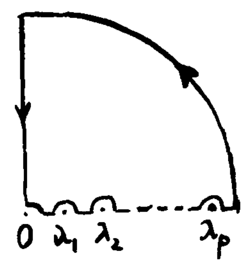

Fig. 5

(ii) a series of small semi-circles γ_s $(s = 1, 2, \ldots, p)$ with equations $|z_s - \lambda_s| = \delta_s$;

(iii) a large circular quadrant $|z| = R$, $0 \leq \arg z \leq \frac{1}{2}\pi$;

(iv) the positive imaginary axis from $z = Re^{\frac{1}{2}i\pi}$ to $z = \delta e^{\frac{1}{2}i\pi}$;

(v) a small circular quadrant $|z| = \delta$, $0 \leq \arg z \leq \frac{1}{2}\pi$.

In general the point $z = 0$ is a branch point of $F(z)$. Each branch is, however, an analytic function and therefore this presents no difficulties; we choose that branch for which $\mathrm{Re}\left\{F(z)\right\} = J_\alpha(\rho z)\,J_\beta(t\,z)z^{1+\gamma}$ for real values of z. Using standard procedures in the calculus of residues we can easily show that

$$S_{\nu,\alpha,\beta,\gamma}(\rho, t; a) = \int_0^\infty J_\alpha(\rho x) J\,(tx) x^{\gamma+1}\,dx + \frac{2}{\pi}\sin\left\{\tfrac{1}{2}(\alpha+\beta+\gamma-2\nu)\right\} K_{\nu,\alpha,\beta,\gamma}(\rho, t; a) \tag{3.23}$$

where the function $K_{\nu,\alpha,\beta,\gamma}(\rho, t; a)$ is the integral defined above.

In a similar way we can obtain an integral representation of the series $S^*_{\nu,H,\beta,\gamma,\delta}(u, v)$ in terms of the integral $K^*_{\nu,H,\beta,\gamma,\delta}(u, v)$ by considering the contour integral

$$\int_C \phi(z) J_\beta(u\,z)\,J_\gamma(v\,z) z^{\delta+1}\,dz$$

where

$$\phi(z) = \frac{z\,J_\nu'(z) + i Y_\nu'(z) + H\,J_\nu(z) + i Y_\nu(z)}{2J_\nu'(z) + H\,J_\nu(z)}$$

and C is the same contour as before, except that now the λ's are the positive roots of equation (3.21). We find that

$$S^{*}_{\nu, H, \beta, \gamma, \delta}(u, v) = \int_{q}^{\infty} J_{\beta}(ux)\, J_{\gamma}(vx)\, x^{\delta+1}\, dx$$

$$+ \frac{2}{\pi} \sin\left\{\tfrac{1}{2}(\delta + \beta + \gamma - 2\nu)\right\} K^{*}_{\nu, H, \beta, \gamma, \delta}(u, v) \qquad (3.24)$$

We see that the expressions for S and S^{*} involve the Weber-Schaftheitlin integral

$$\int_{0}^{\infty} x^{-\lambda}\, J_{\alpha}(\rho x) J_{\beta}(tx)\, dx$$

in which ρ and t are supposed to be positive, and where, to ensure convergence

$$\mathrm{Re}(\alpha + \beta + 1) > \mathrm{Re}(\lambda) > -1, \quad \rho \neq t,$$

$$\mathrm{Re}(\alpha + \beta + 1) > \mathrm{Re}(\lambda) > 0, \quad \rho = t.$$

The evaluation of this integral is a long and complicated affair, for the details of which the reader is referred to pp.398-404 of Watson (1944). We shall merely state the relevant results.

It has been shown by Sonine and Schaftheitlin that the integral

$$\int_{0}^{\infty} x^{\alpha+\beta-\gamma} J_{\alpha-\beta}(ax)\, J_{\gamma-1}(bx)\, dx \qquad (3.25)$$

has the value

$$\frac{b^{\gamma-1}\Gamma(\alpha)}{2^{\gamma-\alpha-\beta} a^{\alpha+\beta}\Gamma(\gamma)\Gamma(1-\beta)}\, {}_2F_1\left(\alpha, \beta; \gamma; \frac{b^2}{a^2}\right) \qquad (3.26)$$

when b < a and the value

$$\frac{a^{\alpha-\beta}\, \Gamma(\alpha)}{2^{\gamma-\alpha-\beta} b^{2\alpha-\gamma+1}\Gamma(\gamma-\alpha)\Gamma(\alpha-\beta+1)}\, {}_2F_1\left(\alpha, \alpha-\gamma+1; \alpha-\beta+1; \frac{a^2}{b^2}\right) \qquad (3.27)$$

when b > a. It will be seen from these results that the integral (3.25) is a function of b/a which is not analytic at the point b/a = 1. When b = a, the value of the integral has to be found by a special procedure for the details of which the reader is referred to p.402 of Watson (1944).

Some particular cases are of special interest. If we put $\alpha = \nu + m + 1$, $\beta = 1 - k - m$, $\gamma = \nu + 1$, a = 1, b = ρ, where m is a positive integer, and k and ν are such that the integral converges we have the result

$$I(\nu, m, k, \rho) = \begin{cases} \dfrac{\rho^{\nu}\, \Gamma(\nu + m + 1)}{2^{k-1}\, \Gamma(\nu + 1)\, \Gamma(m + k)}\, {}_2F_1(\nu + m + 1, 1 - k - m; \nu + 1; \rho^2), & \rho < 1, \\ 0 & \rho > 1, \end{cases}$$

where

$$I(\nu, m, k, \rho) = \int_0^\infty u^{1-k} J_{\nu+2m+k}(u) J_\nu(\rho u) du \tag{3.28}$$

Using the transformation

$${}_2F_1(\alpha, \beta, \gamma; z) = (1-z)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; z)$$

(Sneddon, 1956, p.22) and the definition

$$\mathfrak{F}_m(a, b, x) = {}_2F_1(-m, a+m; b; x)$$

of the Jacobi polynomial we obtain the result

$$I(\nu, m, k, \rho) = \begin{cases} \dfrac{\Gamma(\nu+m+1)\rho^\nu(1-\rho^2)^{k-1}}{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)} \mathfrak{F}_m(k+\nu, \nu+1; \rho^2), & 0 < \rho < 1 \\ 0, & \rho > 1 \end{cases} \tag{3.29}$$

To return to Tranter's series s_1 and s_2 defined by equations (3.18), (3.19) we note that it follows from equations (3.18a) and (3.23) that, since m is zero or a positive integer,

$$s_1(\nu, m, k, \rho) = I(\nu, m, k, \rho) \tag{3.30}$$

where $I(\nu, m, k, \rho)$ is defined by equation (3.28) and can be evaluated by means of the formulae (3.29). In a similar way it follows from equations (3.19a), (3.23) that

$$s_2(\nu, m, n, k) = \tfrac{1}{2} a^2 \int_0^\infty x^{-1} J_{\nu+2m+k}(x) J_{\nu+2n+k}(x) dx$$
$$-(-1)^{m+n} \frac{a^2}{\pi} \sin(k\pi) K_{\nu, \nu+2m+k, \nu+2n+k, -2}(1, 1; a) \tag{3.31}$$

Another special case of equation (3.23) which is of interest is

$$S_{\nu,\nu,\nu-\frac{1}{2}p,-\frac{1}{2}p}(\rho, t; a) = \int_0^\infty J_\nu(\rho x) J_{\nu-\frac{1}{2}p}(tx) x^{1-\frac{1}{2}p} dx$$
$$- \frac{2}{\pi} \sin(\tfrac{1}{2}p\pi) K_{\nu,\nu,\nu-\frac{1}{2}p,-\frac{1}{2}p}(\rho, t; a)$$

which, as a result of equation (3.29), can be written in terms of Heaviside's unit function in the form

$$S_{\nu,\,\nu,\,\nu-\frac{1}{2}p,\,-\frac{1}{2}p}(\rho,\,t;\,a)$$

$$= \frac{2^{1-\frac{1}{2}p}\,t^{\nu-\frac{1}{2}p}(\rho^2-t^2)^{\frac{1}{2}p-1}}{\Gamma(\frac{1}{2}p)\rho^{\nu}}\,H(\rho-t) - \frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)K_{\nu,\,\nu,\,\nu-\frac{1}{2}p,\,-\frac{1}{2}p}(\rho,\,t;\,a). \qquad (3.32)$$

In a similar way we can show that

$$S_{\nu,\,\nu+1,\,\nu-\frac{1}{2}p,\,-\frac{1}{2}p-1}(\rho,\,t;\,a)$$

$$= \frac{2^{-\frac{1}{2}p}\,t^{\nu-\frac{1}{2}p}(\rho^2-t^2)^{\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)\rho^{\nu+1}}\,H(\rho-t) - \frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)K_{\nu,\,\nu+1,\,\nu-\frac{1}{2}p,\,-\frac{1}{2}p-1}(\rho,\,t;a). \qquad (3.33)$$

Further results can be obtained by means of the Hankel inversion theorem. If we apply this theorem to equations (3.28) and (3.29) we can deduce that

$$\int_0^1 \rho^{\nu+1}(1-\rho^2)^{k-1}\mathfrak{F}_m(k+\nu,\,\nu+1,\,\rho^2)J_\nu(\rho u)\,d\rho = \frac{2^{k-1}\Gamma(\nu+1)\,\Gamma(m+k)}{\Gamma(\nu+m+1)}u^{-k}J_{\nu+2m+k}(u)$$

which, by a trivial change of variable, we can write in the form

$$\int_0^s x^{\nu+1}(s^2-x^2)^{k-1}\,\mathfrak{F}_m(k+\nu,\,\nu+1,\,x^2/s^2)J_\nu(\xi x)\,dx$$
$$= \frac{2^{k-1}\Gamma(\nu+1)\Gamma(m+k)}{\Gamma(\nu+m+1)}\,s^{\nu+k}\,\xi^{-k}\,J_{\nu+2m+k}(\xi s). \qquad (3.34)$$

If $m = 0$ the Jacobi polynomial reduces to unity and we obtain the simpler relation

$$\int_0^s x^{\nu+1}(s^2-x^2)^{k-1}J_\nu(\xi x)\,dx = 2^{k-1}\,\Gamma(k)s^{\nu+k}\,\xi^{-k}J_{\nu+k}(\xi s). \qquad (3.35)$$

We also require to obtain integral representations of two trigonometrical series involving Bessel functions.

By integrating the function

$$\operatorname{cosec}(\pi z)e^{i\pi z}J_0(uz)\sin(xz),$$

round the contour Γ which consists of the positive real axis, the positive imaginary axis and the arc, in the first quadrant, of the circle $|z| = R$, with large $\underline{R}$, and with indentations around the points $z = 0, 1, 2, \ldots, n, \ldots$ we can easily show that

$$\sum_{n=1}^{\infty} J_0(nu)\sin(nx) = \int_0^\infty J_0(tu)\sin(tx)dt - \int_0^\infty \frac{e^{-\pi y}}{\sinh(\pi y)} I_0(uy)\sinh(xy)dy$$

Also it is well-known (Watson, 1944, p.405) that

$$\int_0^\infty J_0(tu)\sin(xt)dt = (x^2 - u^2)^{-\frac{1}{2}} H(x - u)$$

where $H(x)$ is Heaviside's unit function. Hence we have relation

$$\sum_{n=1}^{\infty} J_0(nu)\sin(nx) = \frac{H(x-u)}{\sqrt{(x^2 - u^2)}} - \int_0^\infty \frac{\sinh(xy)}{\sinh(\pi y)} e^{-\pi y} I_0(uy)dy \tag{3.36}$$

Similarly, by integrating the function

$$\operatorname{cosec}(\pi z)e^{i\pi z} J_1(uz)\cos(xz)$$

round the same contour Γ we can show that

$$\sum_{n=1}^{\infty} J_1(nu)\cos(nx) = \int_0^\infty J_1(tu)\cos(xt)dt - \int_0^\infty \frac{\cosh(xy)}{\sinh(\pi y)} e^{-\pi y} I_1(uy)dy$$

from which it follows that

$$\sum_{n=1}^{\infty} J_1(nu)\cos(nx) = \frac{1}{u} - \frac{xH(x-u)}{\sqrt{(x^2 - u^2)}} - \int_0^\infty \frac{\cosh(xy)}{\sinh(\pi y)} e^{-\pi y} I_1(uy)dy. \tag{3.37}$$

3.3. Some Integral Equations.

In solving dual series relations we sometimes have to solve an integral equation of the type

$$\int_a^x \frac{h(t)dt}{\left\{f(x) - f(t)\right\}^\alpha} = g(x), \qquad a < x < b \tag{3.38}$$

where $0 < \alpha < 1$ and $f(t)$ is a strictly monotonic increasing function in (a, b). We shall give a solution due to Srivastav (1963c).

Consider the integral

$$\int_a^x \frac{f'(u)g(x)du}{\left\{f(x) - f(u)\right\}^{1-\alpha}} .$$

If we substitute the expression for g(u) given by equation (3.38) and interchange the order of the integrations we find that the integral is equal to

$$\int_a^x h(t)dt \int_t^x \frac{f'(u)du}{\left\{f(u) - f(t)\right\}^{\alpha}\left\{f(x) - f(u)\right\}^{1-\alpha}} .$$

The inner integral is easily shown to have the value $B(\alpha, 1-\alpha) = \pi \operatorname{cosech} \pi\alpha$. It therefore follows that the integral has the value

$$\pi \operatorname{cosec} \pi\alpha \int_a^x h(t)dt$$

and hence that the required solution is

$$h(t) = \frac{\sin \pi\alpha}{\pi} \frac{d}{dt} \int_a^t \frac{f'(u)g(u)du}{\left\{f(t) - f(u)\right\}^{1-\alpha}} . \tag{3.39}$$

By a similar method it can be shown that the integral equation

$$\int_x^b \frac{h(t)dt}{\left\{f(t) - f(x)\right\}^{\alpha}} = g(x), \quad a < x < b \tag{3.40}$$

where $0 < \alpha < 1$ and f(t) is monotonic increasing in (a, b) has solution

$$h(t) = -\frac{\sin \pi\alpha}{\pi} \frac{d}{dt} \int_t^b \frac{f'(u)g(u)du}{\left\{f(u) - f(t)\right\}^{1-\alpha}} \tag{3.41}$$

Two special cases are of particular interest and we shall consider these now. If $f(u) = -\cos u$, $f'(u) = \sin u$ and $\alpha = \frac{1}{2}$ and we find that the integral equation

$$\int_a^x \frac{h(t)dt}{\sqrt{(\cos t - \cos x)}} = g(x), \qquad a < x < b \tag{3.42a}$$

has solution

$$h(t) = \frac{1}{\pi} \frac{d}{dt} \int_a^t \frac{\sin u \, g(u)du}{\sqrt{(\cos u - \cos t)}}, \quad a < t < b \tag{3.42b}$$

and the integral equation

$$\int_x^b \frac{h(t)dt}{\sqrt{\left\{\cos x - \cos t\right)}} = g(x), \quad a < x < b \tag{3.43a}$$

has solution

$$h(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^b \frac{\sin u\, g(u)du}{\sqrt{(\cos t - \cos u)}}, \quad a < t < b. \tag{3.43b}$$

The classic case is the one in which $f(u) = u^2$, $f'(u) = 2u$. In this instance we find that the integral equation

$$\int_a^x \frac{h(t)dt}{(x^2 - t^2)^\alpha} = g(x), \quad 0 < \alpha < 1, \quad a < x < b \tag{3.44a}$$

has solution

$$h(t) = \frac{2\sin\pi\alpha}{\pi}\frac{d}{dt}\int_a^t \frac{u\, g(u)du}{(t^2 - u^2)^{1-\alpha}}, \quad a < t < b. \tag{3.44b}$$

and that the integral equation

$$\int_x^b \frac{h(t)dt}{(t^2 - x^2)^\alpha} = g(x), \quad 0 < \alpha < 1, \quad a < x < b \tag{3.45a}$$

has solution

$$h(t) = -\frac{2\sin\pi\alpha}{\pi}\frac{d}{dt}\int_t^b \frac{u\, g(u)du}{(u^2 - t^2)^{1-\alpha}}, \quad 0 < \alpha < 1, \quad a < t < b. \tag{3.45b}$$

3.4. Some results on Associated Legendre Polynomials.

In the discussion of dual equations involving Fourier-Legendre series of the type (2.52), (2.53) we require certain properties of associated Legendre polynomials. We list these now.

Ferrers' associated Legendre function of the first kind is defined by the equation

$$T^m_{m+n}(x) = (-1)^m (1 - x^2)^{\frac{1}{2}m} \frac{d^m}{dx^m} P_{m+n}(x), \quad (-1 < x < 1), \tag{3.46}$$

m and n being zero or positive integers and $P_r(x)$ denoting the Legendre polynomial of degree r, and by the relation

$$T^{-m}_{m+n}(x) = (-1)^m \frac{\Gamma(n+1)}{\Gamma(n+2m+1)} T^m_{m+n}(x) \tag{3.47}$$

(MacRobert, 1947, pp.125 and 328).

If we assume that the expansion

$$f(\theta) = \sum_{n=0}^{\infty} (2n + 2m + 1)c_n T^{-m}_{m+n}(\cos\theta) \tag{3.48}$$

is valid for $0 \leqslant \theta \leqslant \pi$ and that it can be integrated term by term, then, using standard integrals involving associated Legendre polynomials, we can easily show that the coefficients are given by the formula

$$c_n = \tfrac{1}{2}(-1)^m \int_0^{\pi} f(x) T^m_{m+n}(\cos x) \sin x \, dx \tag{3.49}$$

Using the results of §3.3 we can obtain an integral representation of the associated Legendre polynomial. If we write

$$T^{-m}_{m+n}(\cos\theta) = \cot^m \tfrac{1}{2}\theta \int_0^{\theta} \frac{R^m_{m+n}(u)\,du}{\sqrt{(\cos\theta - \cos\theta)}}, \quad 0 < \theta < \pi, \tag{3.50}$$

then it follows from equations (3.42a and b) that

$$R^m_{m+n}(u) = \frac{1}{\pi}\frac{d}{du}\int_0^u \frac{\sin x \tan^m \tfrac{1}{2}x \, T^{-m}_{m+n}(\cos x)\,dx}{\sqrt{(\cos x - \cos u)}}.$$

The integral on the right is difficult to evaluate but it has been shown by Collins (1961) that

$$R^m_{m+n}(u) = \frac{(-1)^m 2^{m+\frac{1}{2}} \Gamma(n+1)}{\pi \Gamma(n+2n+1)} \sin^{2m}(\tfrac{1}{2}u) \cos(\tfrac{1}{2}u) \left(\frac{1}{\sin u}\frac{d}{du}\right)^m \frac{\cos(n+m+\tfrac{1}{2})u}{\cos\tfrac{1}{2}u}. \tag{3.51}$$

When $m = 0$, this gives $R^0_n(u) = R_n(u)$ where

$$R_n(u) = \frac{\sqrt{2}}{\pi} \cos(n + \tfrac{1}{2})u. \tag{3.52}$$

If we insert this expression into equation (3.50) we obtain the Mehler-Dirichlet integral for $P_n(\cos\theta)$. Similarly, by putting $m = 1$ in equation (3.50) we find that

$$R^1_{n+1}(u) = \frac{1}{\sqrt{2}\,\pi} \sec(\tfrac{1}{2}u) \tan(\tfrac{1}{2}u) \left\{ \frac{\sin(n+1)u}{n+1} + \frac{\sin(n+2)u}{n+2} \right\} \tag{3.53}$$

4. Dual Relations involving Fourier-Bessel Series.

We shall begin by considering the pair of dual series relations

$$\sum_{n=1}^{\infty} \lambda_n^{-p} a_n J_\nu(\rho\lambda_n) = f_1(\rho), \quad 0 \leqslant \rho < 1, \tag{4.1}$$

$$\sum_{n=1}^{\infty} a_n J_\nu(\rho\lambda_n) = f_2(\rho), \quad 1 < \rho \leq a, \tag{4.2}$$

where p and ν are real constants satisfying $-1 \leq p \leq 1$, $\nu > 0$, $\{\lambda_n\}$ are the positive zeros of the Bessel function $J_\nu(a\lambda)$. The functions $f_1(\rho)$ and $f_2(\rho)$ are prescribed and the problem is to determine the sequence of constants $\{a_n\}$. This pair of equations is an immediate generalization of the pairs of equations (2.7), (2.8); (2.20), (2.21).

We split the solution of this problem into two parts by considering two special problems the solutions to which can be combined to give the solution of the general problem:

Problem (a): This is the special problem which occurs when $f_2(\rho) \equiv 0$ in which case the equations reduce to

$$\sum_{n=1}^{\infty} \lambda_n^{-p} a_n J_\nu(\rho\lambda_n) = f_1(\rho), \quad 0 \leq \rho < 1 \tag{4.3}$$

$$\sum_{n=1}^{\infty} a_n J_\nu(\rho\lambda_n) = 0, \quad 1 < \rho \leq a. \tag{4.4}$$

Problem (b): This is the special problem corresponding to the case in which $f_1(\rho) \equiv 0$ and the equations reduce to

$$\sum_{n=1}^{\infty} \lambda_n^{-p} a_n J_\nu(\rho\lambda_n) = 0, \quad 0 \leq \rho < 1 \tag{4.5}$$

$$\sum_{n=1}^{\infty} a_n J_\nu(\rho\lambda_n) = f_2(\rho), \quad 1 < \rho \leq a. \tag{4.6}$$

Problem (a) was first considered by Cooke and Tranter (1959) using a method similar to Tranter's method of solution of dual integral equations. We shall begin with an account of this method and then give an account of an alternative method due to Sneddon and Srivastav (1963) in which the solution to the problem is reduced to that of a Fredholm integral equation of the second kind.

4.1. The Cooke-Tranter Solution of Problem (a).

We need not restrict ν to $\nu \geq 0$ but we assume that ν is not a negative integer and that $\nu > -1 + \frac{1}{2}p$.

If we put $k = 1 - \frac{1}{2}p$ in equations (3.30) and (3.29) we find that

$$\sum_{n=1}^{\infty} \frac{J_{\nu+2m+1-\frac{1}{2}p}(\lambda_n) J_\nu(\lambda_n \rho)}{\lambda_n^{1-\frac{1}{2}p} J^2_{\nu+1}(\lambda_n a)} = 0, \quad (1 < \rho \leq a) \tag{4.7}$$

so that if we take

$$a_n = \frac{1}{\lambda_n^{1-\frac{1}{2}p} J^2_{\nu+1}(\lambda_n a)} \sum_{m=0}^{\infty} b_m J_{\nu+2m+1-\frac{1}{2}p}(\lambda_n) \tag{4.8}$$

the equation (4.4) is automatically satisfied.

The coefficients $\left\{b_m\right\}$ have now to be chosen such that a_n as given by equation (4.8) satisfies equation (4.3). Substituting the expression (4.8) for a_n into equation (4.3) and interchanging the order of the summations

$$\sum_{m=0}^{\infty} b_m \sum_{n=1}^{\infty} \frac{J_{\nu+2m+1-\frac{1}{2}p}(\lambda_n) J_\nu(\lambda_n \rho)}{\lambda_n^{1+\frac{1}{2}p} J^2_{\nu+1}(\lambda_n a)} = f_1(\rho), \quad (0 < \rho < 1). \tag{4.9}$$

Now, by equation (3.34), if <u>s</u> is zero or a positive integer

$$\frac{J_{\nu+2s+1-\frac{1}{2}p}(\lambda_n)}{\lambda_n^{1-\frac{1}{2}p}} = \frac{2^{\frac{1}{2}p}\Gamma(\nu+s+1)}{\Gamma(\nu+1)\Gamma(s+1-\frac{1}{2}p)} \int_0^1 \rho^{\nu+1}(1-\rho^2)^{-\frac{1}{2}p} \times$$

$$\times \mathfrak{F}_s(1-\tfrac{1}{2}p+\nu, \nu+1, \rho^2) J_\nu(\lambda_n \rho)\, d\rho, \tag{4.10}$$

where $\mathfrak{F}_s$ is a Jacobi polynomial. Hence if we multiply equation (4.9) by $\rho^{\nu+1}(1-\rho^2)^{-\frac{1}{2}p}\, \mathfrak{F}_s(1-\frac{1}{2}p+\nu, \nu+1, \rho^2)$, integrate with respect to ρ from 0 to 1, and interchange the order of integration and summation and make use of equation (4.10), we find that

$$\sum_{m=0}^{\infty} b_m B_m(\nu, s, p) = E(\nu, s, p), \tag{4.11}$$

where we have written

$$E(\nu\ ,\ s,\ p) = \frac{2^{\frac{1}{2}p}\Gamma(\nu + s + 1)}{\Gamma(\nu + 1)\Gamma(s + 1 - \frac{1}{2}p)} \int_0^1 \rho^{\nu+1} (1 - \rho^2)^{-\frac{1}{2}p} \times$$

$$\times \mathfrak{F}_s(1 - \tfrac{1}{2}p + \nu,\ \nu + 1, \rho^2) f_1(\rho) d\rho, \tag{4.12}$$

$$B_m(\nu\ ,\ s,\ p) = \sum_{n=1}^{\infty} \frac{J_{\nu + 2m + 1 - \frac{1}{2}p}(\lambda_n) J_{\nu + 2s + 1 - \frac{1}{2}p}(\lambda_n)}{\lambda_n^2 J_{\nu+1}^2 (\lambda_n a)} . \tag{4.13}$$

Equation (4.11) with $s = 0, 1, 2, 3, \ldots$ provides a set of algebraic equations for the determination of the coefficients $\{b_m\}$. Once the values of the b_m's have been found, the coefficients $\{a_n\}$ can be evaluated by means of equation (4.8). The set of equations (4.11) can, under certain circumstances, be solved by an iterative process as follows:

Since ν is not a negative integer and $\nu > -1 + \frac{1}{2}p$ we know from equation (3.31) that, for all values of m and s under discussion here, we have the relation

$$\frac{2}{a^2} B_m(\nu\ ,\ s,\ p) = \frac{\delta_{m,s}}{2\nu + 4s + 2 - p} - \frac{L_{m,s}^{(\nu\ ,\ p)}}{2\nu + 4s + 2 - p}, \tag{4.14}$$

where $\delta_{m,s}$ is the Kronecker delta and $L_{m,s}^{(\nu,\ p)}$ denotes the integral

$$L_{m,s}^{(\nu\ ,\ p)} = (-1)^{m+s}(2\nu + 4s + 2 - p)\frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)\int_0^{\infty} \frac{K_\nu(t)}{tI_\nu(t)} I_{\nu+2m+1-\frac{1}{2}p}\left(\frac{t}{a}\right) \times$$

$$\times\ I_{\nu+2s+1-\frac{1}{2}p}\left(\frac{t}{a}\right) dt \tag{4.15}$$

Substituting from equation (4.14) into equation (4.11), we find that the algebraic equations to determine the coefficients $\{b_m\}$ become

$$b_s - \sum_{m=0}^{\infty} b_m L_{m,s}^{(\nu\ ,\ p)} = \frac{2}{a^2}(2\nu + 4s + 2 - p)E(\nu\ ,\ s,\ p),\ (s = 0, 1, 2, \ldots) \tag{4.16}$$

The iterative solution of this set of linear equations (which can be shown to converge if a is large enough - although it is difficult to determine precise limits for this) is given by

$$b_s = \sum_{r=0}^{\infty} b_s^{(r)}\ ,\quad (s = 0, 1, 2, \ldots) \tag{4.17}$$

where

$$b_s^{(o)} = \frac{2}{a^2}(2\nu + 4s + 2 - p)E(\nu, s, p) \qquad (4.18a)$$

and

$$b_s^{(r)} = \sum_{m=o}^{\infty} L_{m,s}^{(\nu, p)} b_m^{(r-1)}, \quad (r = 1, 2, 3, \ldots). \qquad (4.18b)$$

The determination of the coefficients a_n of the pair of dual series relations (4.3) and (4.4) is contained in equations (4.8), (4.17), (4.18) and (4.15); in a practical problem, the chief difficulty would lie in the computation of the integrals $L_{m,s}^{(\nu, p)}$

4.2. The Reduction of Problem (a) to an Integral Equation.

We consider now the method of Sneddon and Srivastav (1963) by means of which the solution of Problem (a) can be reduced to that of a Fredholm integral equation of the second kind. The cases $\nu = 0$, $\nu > 0$ have to be treated separately and separate consideration is required for $p > 0$ and $p < 0$.

Problem (a) with $0 < p \leq 1$, $\nu > 0$:

Suppose that for $0 \leq \rho < 1$,

$$\sum_{n=1}^{\infty} a_n J_\nu(\rho \lambda_n) = \alpha(\rho); \qquad (4.19)$$

then the problem will be solved if we can determine $\alpha(\rho)$, for, by the Fourier-Bessel expansion theorem,

$$a_n = \frac{2}{a^2 J_{\nu+1}^2(a\lambda_n)} \int_0^1 u\, \alpha(u) J_\nu(u\lambda_n)\, du. \qquad (4.20)$$

We now represent the function $\alpha(\rho)$ in terms of an auxiliary function $g(t)$ through the formula

$$\alpha(\rho) = -\rho^{\nu-1} \frac{\partial}{\partial\rho} \int_\rho^1 g(t)(t^2 - \rho^2)^{-\frac{1}{2}p}\, dt \qquad (4.21)$$

and reduce the problem to that of determining $g(t)$. Substituting from equation (4.21) into equation (4.20), interchanging the order of the integrations and making use of the formula (3.35) we find that

$$a_n = \frac{2^{1-\frac{1}{2}p}\Gamma(1-\frac{1}{2}p)\lambda_n^{\frac{1}{2}p}}{a^2 J_{\nu+1}^2(a\lambda_n)} \int_0^1 t^{\nu-\frac{1}{2}p} J_{\nu-\frac{1}{2}p}(t\lambda_n)\, g(t)\, dt. \qquad (4.22)$$

Substituting these values for the coefficients a_r into equation (4.3) and interchanging the order of integration and summation we get

$$2^{-\frac{1}{2}p}\Gamma(1-\tfrac{1}{2}p)\int_0^1 t^{\nu-\frac{1}{2}p}g(t)S_{\nu,\nu,\nu-\frac{1}{2}p,-\frac{1}{2}p}(\rho,t;a)dt = f_1(\rho), \quad 0 \leqslant \rho < 1, \qquad (4.23)$$

where $S_{\nu,\nu,\nu-\frac{1}{2}p,-\frac{1}{2}p}(\rho, t; a)$ is defined by equation (3.17). Making use of the formula (3.32) we find that this equation may be written in the form

$$I_{\nu-\frac{1}{2}p-\frac{1}{2},\frac{1}{2}p}\left\{g(t);\rho\right\} = \frac{2^p}{\Gamma(1-\frac{1}{2}p)}\rho^{1-\nu}f_1(\rho)$$

$$+\frac{2^{1+\frac{1}{2}p}}{\pi}\sin\left(\tfrac{1}{2}p\pi\right)\int_0^1 u^{\nu-\frac{1}{2}p}g(u)\,{}^{1-\nu}K_{\nu,\nu,\nu-\frac{1}{2}p,-\frac{1}{2}p}(\rho,u,a)du$$

where K is the integral defined in § 3.1. Using the formula (3.6) we obtain the relation

$$g(t) = \frac{2^p}{\Gamma(1-\frac{1}{2}p)}I_{\nu-\frac{1}{2},-\frac{1}{2}p}\left\{\rho^{1-\nu}f_1(\rho);\ t\right\}$$

$$+\frac{2^{1+\frac{1}{2}p}}{\pi}\sin\ \tfrac{1}{2}p\pi\int_0^1 u^{\nu-\frac{1}{2}p}g(u)I_{\nu-\frac{1}{2}p,-\frac{1}{2}p}\left\{{}^{1-\nu}K_{\nu,\nu,\nu-\frac{1}{2}p,-\frac{1}{2}p}(\rho,u,a);\quad t\right\}du$$

Making use of equations (3.5) and (3.10) we see that this equation reduces to

$$g(t) = \frac{2^p t^{1-\nu}}{\Gamma(1-\frac{1}{2}p)}I_{\frac{1}{2}\nu,-\frac{1}{2}p}\left\{f_1(\rho);\ t\right\}$$

$$+\frac{2}{\pi}\sin\ \tfrac{1}{2}p\pi\ \ t^{1+\frac{1}{2}p-\nu}\int_0^1 u^{\nu-\frac{1}{2}p}g(u)K_{\nu,\ \nu-\frac{1}{2}p,\ -\frac{1}{2}p,0}(t,u)du, \qquad (4.24)$$

which is a Fredholm integral of the second kind for the determination of the function $g(t)$.

<u>Problem</u> (<u>a</u>) <u>with</u> $-1 \leqslant p < 0,\ \nu > 0.$

In this case we multiply both sides of equation (4.3) by $\rho^{\nu+1}$ and integrate with respect to ρ from 0 to ρ to obtain the relation

$$\sum_{n=1}^{\infty}\lambda_n^{-p-1}a_nJ_{\nu+1}(\rho\lambda_n) = F_1(\rho), \quad 0 \leqslant \rho < 1 \qquad (4.25)$$

where

$$F_1(\rho) = \rho^{-\nu-1} \int_0^\rho u^{\nu+1} f_1(u)du.$$

Again we assume that the coefficients $\left\{a_n\right\}$ can be represented by means of the equations (4.20) and (4.21) and hence by equation (4.22). Substituting from equation (4.22) into equation (4.25) and interchanging the order of summation and integration we find that equation (4.25) is equivalent to the relation

$$2^{-\frac{1}{2}p}\,\Gamma(1-\tfrac{1}{2}p)\int_0^1 t^{\nu-\frac{1}{2}p}\,g(t)S_{\nu,\nu+1,\nu-\frac{1}{2}p,-\frac{1}{2}p-1}(\rho,t,a)dt = F_1(\rho). \tag{4.26}$$

Using equation (3.33) we find that this reduces to

$$I_{\nu-\frac{1}{2}p-\frac{1}{2},\frac{1}{2}p+1}\left\{g(t);\ \rho\right\}$$

$$= \frac{2^{p+1}\rho^{-\nu}F_1(\rho)}{\Gamma(1-\frac{1}{2}p)} + \frac{2^{2+\frac{1}{2}p}}{\pi}\sin(\tfrac{1}{2}p\pi)\rho^{-\nu}\int_0^1 g(u)\,u^{\nu-\frac{1}{2}p}K_{\nu,\nu+1,\nu-\frac{1}{2}p,-\frac{1}{2}p-1}(\rho,u;a)du.$$

With the help of equations (3.5), (3.6) and (3.10) we can easily show that this reduces to

$$g(t) = \frac{2^{p+1}t^{-2\nu-1}}{\Gamma(1-\frac{1}{2}p)}\,I_{0,-\frac{1}{2}p-1}\left\{\int_0^\rho f(u)du;\ t\right\}$$

$$+\frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)\int_0^1 g(u)u^{\nu-\frac{1}{2}p}\,t^{\frac{1}{2}p-\nu+1}K_{\nu,\nu-\frac{1}{2}p,\nu-\frac{1}{2}p,0}(t,u;a)du \tag{4.27}$$

which is again a Fredholm equation of the second kind.

<u>Problem</u> (<u>a</u>) <u>with</u> $-1 \leqslant \underline{p} < 0$, $\nu = 0$.

The procedure is essentially the same as that outlined above and the final results we obtain could be obtained merely by letting ν tend to zero. However a slight modification of the calculations is necessary when $\nu = 0$ and the method breaks down for $0 < p < 1$.

For $-1 \leqslant p < 0$ we assume as before that

$$\sum_{n=1}^{\infty} a_n J_0(\rho\lambda_n) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_\rho^1 g(t)(t^2-\rho^2)^{-\frac{1}{2}p}dt, \qquad 0 \leqslant \rho < 1$$

and this leads to the relation

$$a_n = -\frac{2}{a^2 J_1^2(a\lambda_n)}\int_0^1 J_0(\rho\lambda_n)\left\{\frac{\partial}{\partial\rho}\int_\rho^1 g(t)(t^2-\rho^2)^{-\frac{1}{2}p}dt\right\}d\rho$$

which, as the result of an integration by parts, can be written in the form

$$a_n = \frac{2}{a^2 J_1^2(a\lambda_n)} \left\{ \int_0^1 t^{-p} g(t)dt - \lambda_n \int_0^1 g(t)dt \int_0^t J_1(\rho\lambda_n)(t^2-\rho^2)^{-\frac{1}{2}p} d\rho . \right.$$

Using the fact that

$$\int_0^t J_1(\rho\lambda_n)(t^2-\rho^2)^{-\frac{1}{2}p} d\rho = \lambda_n^{-1} t^{-p} + p\lambda_n 2^{-\frac{1}{2}p-1} (-\tfrac{1}{2}p)\lambda_n^{\frac{1}{2}p} t^{-\frac{1}{2}p} J_{-\frac{1}{2}p}(t\lambda_n)$$

we find that

$$a_n = \frac{2^{1-\frac{1}{2}p}\lambda_n^{\frac{1}{2}p}\Gamma(1-\frac{1}{2}p)}{a^2 J_1^2(a\lambda_n)} \int_0^1 t^{-\frac{1}{2}p} g(t)\, J_{-\frac{1}{2}p}(t\lambda_n)dt \tag{4.28}$$

This expression for a_n is identical with the one we should obtain by putting $\nu = 0$ in (4.22). The result of the analysis is the same as that given by letting $\nu \longrightarrow 0$ in equation (4.27) so that we obtain the integral equation

$$g(t) = \frac{2^{p+1}}{t\,\Gamma(1-\frac{1}{2}p)} I_{0,\,-\frac{1}{2}p-1} \left\{ \int_0^\rho f(u)du;\ t \right\}$$

$$+ \frac{2}{\pi}\sin(\tfrac{1}{2}p\pi) \int_0^1 g(u)\, u^{-\frac{1}{2}p}\, t^{1+\frac{1}{2}p} K_{0,\,-\frac{1}{2}p,\,-\frac{1}{2}p,\,0}(t, u; a)du \tag{4.29}$$

for the determination of the function $g(t)$.

<u>Problem</u> (<u>a</u>) <u>with</u> $\rho = 1$, $\nu = 0$.

To solve the dual series equations

$$\sum_{n=1}^{\infty} \lambda_n^{-1} a_n J_0(\rho\lambda_n) = f_1(\rho), \qquad 0 \leqslant \rho < 1, \tag{4.30}$$

$$\sum_{n=1}^{\infty} a_n J_0(\rho\lambda_n) = 0, \qquad 1 < \rho \leqslant a, \tag{4.31}$$

we set

$$\sum_{n=1}^{\infty} a_n J_0(\rho\lambda_n) = \alpha(\rho) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_\rho^1 \frac{th(t)dt}{\sqrt{(t^2-\rho^2)}} \tag{4.32}$$

for $0 \leq \rho < 1$. Then making use of the formula for determining the coefficients in a Fourier-Bessel expansion and using the rule for integration by parts to evaluate the relevant integral we find that

$$a_n = \frac{2}{a^2 J_1^2(a\lambda_n)}\left\{\int_0^1 h(t)dt + \lambda_n \int_0^1 th(t)dt \int_0^t \frac{J_1(\rho\lambda_n)d\rho}{\sqrt{(t^2-\rho^2)}}\right\}.$$

The integral involving the Bessel function is readily evaluated and we find that

$$a_n = \frac{2}{a^2 J_1^2(a\lambda_n)}\int_0^1 h(t)\cos(\lambda_n t)dt. \qquad (4.33)$$

If we now substitute this value of a_n in equation (4.30) and interchange the order of summation and integration we obtain the relation

$$\sqrt{(\tfrac{1}{2}\pi)}\int_0^1 t^{\frac{1}{2}} h(t)\, S_{0,0,-\frac{1}{2},-\frac{1}{2}}(\rho, t; a)dt = f_1(\rho), \quad 0 \leq \rho < 1$$

which because of (3.32) with $\nu = 0$, $\rho = 1$ can be written in the form

$$I_{-\frac{1}{2},\frac{1}{2}}\left\{h(t); \rho\right\} = \frac{2}{\sqrt{\pi}} f_1(\rho) + \frac{2\sqrt{2}}{\pi}\int_0^1 th(t)K_{0,0,-\frac{1}{2},-\frac{1}{2}}(\rho, t; a)dt.$$

Applying the operator $I^{-1}_{-\frac{1}{2},\frac{1}{2}}$ to both sides of this equation we obtain the Fredholm equation

$$h(t) = \frac{2t}{\pi}\int_0^t \frac{f_1'(\rho)d\rho}{\sqrt{(t^2-\rho^2)}} + \frac{4}{\pi^2}\int_0^1 h(u)K_{0,-\frac{1}{2},-\frac{1}{2},-1}(t, u; a)dt \qquad (4.34)$$

for the determination of the function $h(t)$ in terms of which a_n can be calculated by means of equation (4.33) and the function $\alpha(\rho)$ by equation (4.32).

4.3. The Reduction of Problem (b) to an Integral Equation.

We now turn our attention to Problem (b). We begin by considering the case in which the parameter p occurring in the dual series equations

$$\sum_{n=1}^{\infty} \lambda_n^{-p} a_n J_\nu(\rho\lambda_n) = 0, \qquad 0 \leq \rho < 1, \qquad (4.35)$$

$$\sum_{n=1}^{\infty} a_n J_\nu(\rho\lambda_n) = f(\rho), \qquad 1 < \rho \leq a, \qquad (4.36)$$

satisfies the inequality $0 < p \leq 1$. We shall assume that $\nu > 0$.

When $0 \leq \rho < 1$, we make the representation

$$\sum_{n=1}^{\infty} a_n J_\nu(\rho \lambda_n) = -\rho^{\nu-1} \frac{\partial}{\partial \rho} \int_\rho^1 g(t)(t^2 - \rho^2)^{-\frac{1}{2}p}\, dt. \tag{4.37}$$

Using the Fourier-Bessel inversion theorem and performing an integration by parts in the integral involved, we find that

$$a_n = \frac{2^{1-\frac{1}{2}p}\Gamma(1-\frac{1}{2}p)}{a^2 J^2_{\nu+1}(a\lambda_n)} \lambda_n^{\frac{1}{2}p} \int_0^1 g(t) t^{\nu-\frac{1}{2}p} J_{\nu-\frac{1}{2}p}(t\lambda_n)dt$$

$$+ \frac{2}{a^2 J^2_{\nu+1}(a\lambda_n)} \int_1^a u f(u) J_\nu(u\lambda_n)du. \tag{4.38}$$

If we substitute this expression for a_n into equation (4.35) and interchange the order of the integration and summation, we obtain the relation

$$2^{-\frac{1}{2}p}\Gamma(1-\tfrac{1}{2}p)\int_0^1 g(t)t^{\nu-\frac{1}{2}p} S_{\nu,\,\nu,\,\nu-\frac{1}{2}p,\,-\frac{1}{2}p}(\rho, t;\, a)dt = \psi_1(\rho), \tag{4.39}$$

where

$$\psi(\rho) = -\int_1^a u f(u) S_{\nu,\,\nu,\,\nu,\,-p}(\rho, u;\, a)du.$$

From equations (4.23)-(4.24) we see that the equation (4.39) is equivalent to the Fredholm integral equation

$$g(t) = \frac{2^p t^{1-\nu}}{\Gamma(1-\frac{1}{2}p)} I_{\frac{1}{2}\nu,\,-\frac{1}{2}p}\left\{\psi_1(\rho);\, t\right\}$$

$$+ \frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)t^{1-\nu+\frac{1}{2}p}\int_0^1 u^{\nu-\frac{1}{2}p} g(u) K_{\nu-\frac{1}{2}p,\,\nu-\frac{1}{2}p,\,0}(t, u)du \tag{4.40}$$

for the determination of the function $g(t)$.

The case in which the parameter p satisfies the inequality $-1 \leq p < 0$ can be treated in a similar fashion. Multiplying both sides of equation (4.35) by $\rho^{\nu+1}$ and integrating with respect to ρ from 0 to ρ, we obtain the equation

$$\sum_{n=1}^{\infty} \lambda_n^{-p-1} a_n J_{\nu+1}(\rho\lambda_n) = 0, \quad 0 \leq \rho < 1. \tag{4.41}$$

If we substitute the expression (4.38) for the a_n's on the left hand side of equation (4.41) we find that

$$2^{-\frac{1}{2}p}\Gamma(1-\tfrac{1}{2}p)\int_0^1 g(t)t^{\nu-\frac{1}{2}p}S_{\nu,\nu+1,\nu-\frac{1}{2}p,-\frac{1}{2}p-1}(\rho,t;a)dt = \psi_2(\rho), \tag{4.42}$$

where $0 \leq \rho < 1$ and $\psi_2(\rho)$ is defined by the equation

$$\psi_2(\rho) = -\int_1^a uf(u)S_{\nu,\nu+1,\nu,-p-1}(\rho,u;a)du, \quad 0 \leq \rho < 1.$$

Applying the analysis involved in the derivation of the equations (4.26)- (4.27) to equation (4.42) we find immediately that the solution of the problem is reduced to determining the function g(t) from the Fredholm integral equation

$$g(t) = \frac{2^{p+1}t^{-2\nu-1}}{\Gamma(1-\frac{1}{2}p)} I_{0,-\frac{1}{2}p-1}\left\{\rho^{\nu+1}\psi_2(\rho);\ t\right\}$$

$$+\frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)\int_0^1 g(u)u^{\nu-\frac{1}{2}p}t^{\frac{1}{2}p-\nu+1}K_{\nu-\frac{1}{2}p,\nu-\frac{1}{2}p,0}(t,u;a)du. \tag{4.43}$$

5. Dual Relations involving Dini Series.

If the function $f(x)$ is defined in the closed interval $[0, 1]$, its Dini expansion is, in general,

$$\sum_{m=1}^{\infty} b_m J_m(\lambda_n x) \tag{5.1}$$

where $\{\lambda_n\}$ is the sequence of positive roots (arranged in ascending order of magnitude) of the transcendental equation

$$\lambda J_\nu'(\lambda) + HJ_\nu(\lambda) = 0, \tag{5.2}$$

H and ν being real constants with $\nu \geqslant -\frac{1}{2}$. The coefficients b_m are given by the formula

$$b_m = \frac{2\lambda_m^2}{(\lambda_m^2 - \nu^2 + H^2)J_\nu^2(\lambda_m)} \int_0^1 tf(t)\, J_\nu(t\lambda_m)dt. \tag{5.3}$$

The expansion (5.1) holds if $H + \nu > 0$; if $H + \nu = 0$ an initial term

$$2(\nu + 1)x^\nu \int_0^1 t^{\nu+1} f(t)dt \tag{5.4}$$

has to be added to the series, while if $H + \nu < 0$ the equation (5.2) has two purely imaginary zeros ($\pm i\lambda_0$, say) and an initial term

$$\frac{2\lambda_0^2 I_\nu(\lambda_0 x)}{(\lambda_0^2 + \nu^2 - H^2)I_\nu^2(\lambda_0)} \int_0^1 tf(t)\, I_\nu(t\lambda_0)dt \tag{5.5}$$

has to be added. (Watson, 1944; Chapter XVIII).

We now consider the relations

$$\sum_{n=1}^{\infty} c_n \lambda_n^p J_\nu(\rho\lambda_n) = f_1(\rho), \quad 0 \leqslant \rho < c \tag{5.6}$$

$$\sum_{n=1}^{\infty} c_n\, J_\nu(\rho\lambda_n) = f_2(\rho), \quad c < \rho \leqslant 1. \tag{5.7}$$

Solution in the case $-1 \leqslant p < 0$, $\nu > 0$.

We assume that when $0 \leqslant \rho < c$

$$\sum_{n=1}^{\infty} c_n J_\nu(\rho\lambda_n) = -\rho^{\nu-1}\frac{\partial}{\partial\rho}\int_\rho^c g(t)(t^2-\rho^2)^{\frac{1}{2}p}\,dt.$$

From equation (5.3) it follows that

$$c_n = \frac{2\lambda_n^2}{(\lambda_n^2-\nu^2+H^2)J_\nu^2(\lambda_n)}\left\{\int_c^1 tf_2(t)\,J_\nu(t\lambda_n)dt + 2^{\frac{1}{2}p}\lambda_n^{-\frac{1}{2}p}\Gamma(1+\tfrac{1}{2}p)\times\right.$$

$$\left.\times\int_0^c g(t)t^{\nu+\frac{1}{2}p}J_{\nu+\frac{1}{2}p}(t\lambda_n)dt\right\}. \tag{5.8}$$

If we now substitute this value of c_n in equation (5.6) and interchange the order of integration and summation, we obtain the relation

$$2^{\frac{1}{2}p}\Gamma(1+\tfrac{1}{2}p)\int_0^c t^{\nu+\frac{1}{2}p}g(t)\,S^*_{\nu,H,\nu,\nu+\frac{1}{2}p,\frac{1}{2}p}(\rho,t)dt = \psi_1(\rho),\quad 0\leq\rho<c \tag{5.9}$$

where the function

$$\psi_1(\rho) = f_1(\rho) + \int_c^1 tf_2(t)\,S^*_{\nu,H,\nu,\nu,p}(\rho,t)dt \tag{5.10}$$

is known.

Now from equations (3.24) and (3.29)

$$S^*_{\nu,H,\nu,\nu+\frac{1}{2}p,\frac{1}{2}p}(\rho,t) = \frac{2^{1+\frac{1}{2}p}}{\Gamma(-\frac{1}{2}p)}t^{\nu+\frac{1}{2}p}(\rho^2-t^2)^{-\frac{1}{2}p-1}\rho^{-\nu}H(\rho-t)$$

$$+\frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)\,K^*_{\nu,H,\nu,\nu+\frac{1}{2}p,\frac{1}{2}p}(\rho,t),$$

where $H(x)$ is Heaviside's unit function, so that equation (5.9) can be written in the form

$$I_{\nu-\frac{1}{2},-\frac{1}{2}p}\left[t^p g(t);\rho\right]$$

$$=\frac{2^{-p}}{\Gamma(1+\frac{1}{2}p)}\rho^{1+p-\nu}\left[\psi_1(\rho)-\frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)\int_0^1 K^*_{\nu,H,\nu,\nu+\frac{1}{2}p,\frac{1}{2}p}(\rho,t)g(t)t^{\nu+\frac{1}{2}p}dt\right]$$

which by application of the operator $I^{-1}_{\nu-\frac{1}{2},-\frac{1}{2}p}$ can be written in the form

$$g(t) = \frac{2^{-p}t^{1-\nu}}{\Gamma(1+\frac{1}{2}p)}\left[I_{\frac{1}{2}\nu,\frac{1}{2}p}\left\{f_1(\rho);\ t\right\} + \left(\frac{2}{t}\right)^{\frac{1}{2}p}\int_c^1 uf_2(u)S^*_{\nu,H,\nu+\frac{1}{2}p,\nu,\frac{1}{2}p}(t,u)du\right.$$

$$\left. - \left(\frac{2}{t}\right)^{\frac{1}{2}p}\frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)\int_0^c g(u)u^{\nu+\frac{1}{2}p}K^*_{\nu,H,\nu+\frac{1}{2}p,\nu+\frac{1}{2}p,0}(t,u)du\right]. \quad (5.11)$$

Equation (5.11) is a Fredholm equation of the second kind by means of which we can determine the function $g(t)$ and hence determine the constants $\left\{c_n\right\}$.

<u>Solution in the case $0 < p \leq 1$, $\nu > 0$.</u>

If $0 < p \leq 1$ and $\nu > 0$ the above procedure has to be modified slightly. Multiplying both sides of equation (5.6) by $\rho^{\nu+1}$ and integrating with respect to ρ from 0 to ρ, we see that equation (5.6) is equivalent to the equation

$$\sum_{n=1}^{\infty} c_n\lambda_n^{p-1} J_{\nu+1}(\rho\lambda_n) = \rho^{-\nu-1}\int_0^\rho t^{\nu+1}f_1(t)dt, \quad 0 \leq \rho < c. \quad (5.12)$$

Substituting the expression (5.8) for c into this equation and interchanging the order of integration and summation we obtain the relation

$$2^{\frac{1}{2}p}\Gamma(1+\tfrac{1}{2}p)\int_0^c t^{\nu+\frac{1}{2}p}S^*_{\nu,H,\nu+1,\nu+\frac{1}{2}p,\frac{1}{2}p-1}(\rho,t)g(t)dt = \psi_2(\rho), \quad 0 \leq \rho < c \quad (5.13)$$

where

$$\psi_2(\rho) = \rho^{-\nu-1}\int_0^\rho t^{\nu+1}f_1(t)dt + \int_c^1 tf_2(t)S^*_{\nu,H,\nu+1,\nu,p-1}(\rho,t)dt \quad (5.14)$$

for $0 \leq \rho < c$. Since

$$S^*_{\nu,H,\nu+1,\nu+\frac{1}{2}p,\frac{1}{2}p-1}(\rho,t)$$

$$= \frac{2^{\frac{1}{2}p}t^{\nu+\frac{1}{2}p}(\rho^2-t^2)^{-\frac{1}{2}p}}{\Gamma(1-\frac{1}{2}p)}\rho^{-\nu-1}H(\rho-t) + \frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)K^*_{\nu,H,\nu+1,\nu+\frac{1}{2}p,\frac{1}{2}p-1}(\rho,t) \quad (5.15)$$

it follows that we may write equation (5.13) in the form

$$I_{\nu-\frac{1}{2},1-\frac{1}{2}p}\left\{t^p g(t);\rho\right\}$$

$$= \frac{2^{1-p}\rho^{p-\nu}}{\Gamma(1+\frac{1}{2}p)}\left[\psi_2(\rho) - \frac{2}{\pi}\sin(\tfrac{1}{2}p\pi)\int_0^c t^{\nu+\frac{1}{2}p}K^*_{\nu,H,\nu+1,\nu+\frac{1}{2}p,\frac{1}{2}p-1}(\rho,t)g(t)dt\right]. \quad (5.16)$$

Applying the operator $I^{-1}_{\nu-\frac{1}{2},\,1-\frac{1}{2}p}$ to both sides of this equation and making use of the results (3.6), (3.15) we see that this equation is equivalent to the equation

$$t^p g(t) = \frac{2^{1-p}\, t^{p-\nu}}{\Gamma(1+\frac{1}{2}p)}\Bigg[I_{\frac{1}{2}\nu+\frac{1}{2},\,\frac{1}{2}p-1}\Big\{\psi_2(\rho);\ t\Big\}$$

$$- \frac{2^{\frac{1}{2}p}}{\pi}\sin(\tfrac{1}{2}p\pi)\,t^{1-\nu-\frac{1}{2}p}\int_0^c u^{\nu+\frac{1}{2}p}\, g(u) K^*_{\nu,\,H,\,\nu+\frac{1}{2}p,\,\nu+\frac{1}{2}p,\,0}(t,u)\,du\Bigg].$$

Substituting the expression (5.14) for $\psi_2(\rho)$ we see that this equation is equivalent to the Fredholm integral equation

$$g(t) = \frac{2^{1-p}\, t^{-2\nu-1}}{\Gamma(1+\frac{1}{2}p)}\, I_{0,\,\frac{1}{2}p-1}\Big\{\int_0^\rho u f_1(u)\,du;\ t\Big\}$$

$$+ \frac{2^{-\frac{1}{2}p}\, t^{1-\nu-\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\int_c^1 u f_2(u)\, S^*_{\nu,\,H,\,\nu+\frac{1}{2}p,\,\nu,\,\frac{1}{2}p}(t,u)\,du$$

$$- \frac{2^{1-\frac{1}{2}p}\, t^{1-\nu-\frac{1}{2}p}}{\pi\Gamma(1+\frac{1}{2}p)}\sin(\tfrac{1}{2}p\pi)\int_0^c K^*_{\nu,\,H,\,\nu+\frac{1}{2}p,\,\nu+\frac{1}{2}p,\,0}(t,u)\,u^{\nu+\frac{1}{2}p} g(u)\,du \qquad (5.17)$$

by means of which the function $g(t)$ can be determined.

<u>Solution in the case</u> $0 < \underline{p} \leq 1$, $\nu = 0$, $\underline{H} > 0$.

We now examine the dual series relations

$$\sum_{n=1}^{\infty} c_n \lambda_n^p J_0(\rho\lambda_n) = f_1(\rho), \qquad 0 \leq \rho < c, \qquad (5.18)$$

$$\sum_{n=1}^{\infty} c_n J_0(\rho\lambda_n) = f_2(\rho), \qquad c < \rho \leq 1,$$

for $0 < p \leq 1$, it being assumed that the $\{\lambda_n\}$ are the positive roots of the equation (5.2) with $\underline{H} + \nu > 0$. We make the representation

$$\sum_{n=1}^{\infty} c_n J_0(\rho\lambda_n) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_\rho^c g(t)(t^2-\rho^2)^{\frac{1}{2}p}\,dt, \qquad 0 \leq \rho < c \qquad (5.19)$$

from which it follows, by equation (5.3), that

$$c_n = \frac{2\lambda_n^2}{(\lambda_n^2 + H^2)J_0^2(\lambda_n)}\left[\int_c^1 t f_2(t)J_0(t\lambda_n)dt - \int_0^c J_0(\rho\lambda_n)\left\{\frac{\partial}{\partial\rho}\int_\rho^c g(t)(t^2-\rho^2)^{\frac{1}{2}p}dt\right\}d\rho\right].$$

Now

$$\int_0^c J_0(\rho\lambda_n)\left[\frac{\partial}{\partial\rho}\int_\rho^c g(t)(t^2-\rho^2)^{\frac{1}{2}p}\,dt\right]d\rho$$

$$= -\int_0^c g(t)t^p\,dt + \lambda_n\int_0^c g(t)dt\int_0^t J_1(\rho\lambda_n)(t^2-\rho^2)^{\frac{1}{2}p}\,d\rho$$

and

$$\int_0^t J_1(\rho\lambda_n)(t^2-\rho^2)^{\frac{1}{2}p}\,d\rho = \frac{t^p}{\lambda_n} - \frac{2^{\frac{1}{2}p}\Gamma(1+\frac{1}{2}p)}{\lambda_n^{1+\frac{1}{2}p}}t^{\frac{1}{2}p}J_{\frac{1}{2}p}(t\lambda_n)$$

so that

$$c_n = \frac{2\lambda_n^2}{(\lambda_n^2 + H^2)J_0^2(\lambda_n)}\left[\int_c^1 t f_2(t)\,J_0(t\lambda_n)dt\right.$$

$$\left. + 2^{\frac{1}{2}p}\Gamma(1+\tfrac{1}{2}p)\lambda_n^{-\frac{1}{2}p}\int_0^c t^{\frac{1}{2}p}J_{\frac{1}{2}p}(t\lambda_n)g(t)dt\right]. \tag{5.20}$$

This is exactly the expression we should have obtained for c_n if we had put $\nu = 0$ in the right hand side of equation (5.8) but it is arrived at by a different method; the rest of the analysis follows the same course as that leading to equation (5.17) and we find finally that $g(t)$ is the solution of the Fredholm integral equation

$$g(t) = \frac{2^{1-p}}{t\,\Gamma(1+\frac{1}{2}p)}I_{0,\,\frac{1}{2}p-1}\left\{\int_0^\rho uf_1(u)\,du;\ t\right\}$$

$$+\frac{2^{-\frac{1}{2}p}t^{1-\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\int_c^1 uf_2(u)\,S^*_{0,\,H,\,\frac{1}{2}p,\,0,\,\frac{1}{2}p}(t,u)du$$

$$-\frac{2^{1-\frac{1}{2}p}t^{1-\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\sin(\tfrac{1}{2}p\pi)\int_0^c K^*_{0,\,H,\,\frac{1}{2}p,\,\frac{1}{2}p,\,0}(t,u)u^{\frac{1}{2}p}g(u)du. \tag{5.21}$$

Solution in the case $\nu + H = 0$.

The special case in which $\nu + H = 0$ is particularly important. The dual Dini series then take the form

$$\alpha c_0 \rho^{\nu} + \sum_{n=1}^{\infty} \lambda_n^{p} c_n J_{\nu}(\rho\lambda_n) = f_1(\rho), \quad 0 \leqslant \rho < c, \tag{5.22}$$

$$c_0 \rho^{\nu} + \sum_{n=1}^{\infty} c_n J_{\nu}(\rho\lambda_n) = f_2(\rho), \quad c < \rho \leqslant 1, \tag{5.23}$$

where $-1 \leqslant p \leqslant 1$ and $\lambda_1, \lambda_2, \ldots$ are the positive zeros of $J_{\nu+1}(\lambda)$. The constant α would appear to be indeterminate, but we saw previously (Cf. equations (2.27), (2.28) above) that in some physical problems its value is known ab initio; in other problems its value may be derived, at a later stage in the analysis, from some physical criterion such as that a certain component of stress must always remain finite.

When $0 < p \leqslant 1$ the above analysis can readily be adapted to meet the present situation and we find that everything goes through easily; in the end we find that the solution is given by equations (5.8) and (5.17) if $\nu \neq 0$ and by equations (5.20) and (5.21) if $\nu = 0$ except that in equations (5.17) and (5.21) the functions $K^{*}_{\nu, H, \beta, \gamma, \delta}(u, v)$ and $S^{*}_{\nu, H, \beta, \gamma, \delta}(u, v)$ are replaced by $K_{\nu+1, \beta, \gamma, \delta}(u, v)$ and $S_{\nu+1, \beta, \gamma, \delta}(u, v)$ respectively. In other words when $\nu \neq 0$ the equations have solution

$$c_0 = 2(\nu + 1)\int_c^1 t^{\nu+1} f_2(t)dt + \frac{2\Gamma(\nu+2)\Gamma(\frac{1}{2}p+1)}{\Gamma(\nu+\frac{1}{2}p+1)}\int_0^c u^{2\nu+p} g(u)du \tag{5.24}$$

$$c_n = \frac{2}{J_{\nu}(\lambda_n)}\left\{\int_c^1 t f_2(t) J_{\nu}(\lambda_n t)dt + 2^{\frac{1}{2}p}\lambda_n^{-\frac{1}{2}p}\Gamma(1+\tfrac{1}{2}p)\int_0^c t^{\nu+\frac{1}{2}p} g(t) J_{\nu+\frac{1}{2}p} \times \right.$$
$$\left. \times (t\lambda_n)dt\right\}, \tag{5.25}$$

where $g(t)$ is the solution of the Fredholm equation

$$g(t) = \chi(t) - \int_0^c L(t, u) g(u) du, \tag{5.26}$$

where the free term $\chi(t)$ is defined by the equation

$$\chi(t) = \frac{2^{1-p}\,t^{-2\nu-1}}{\Gamma(1+\frac{1}{2}p)}\, I_{0,\,\frac{1}{2}p-1}\left\{\int_0^{\rho} u f_1(u)du;\;\; t\right\}$$

$$+\frac{2^{-\frac{1}{2}p}\,t^{1-\nu-\frac{1}{2}p}}{\Gamma(1+\frac{1}{2}p)}\int_c^1 u f_2(u) S_{\nu+1,\,\nu+\frac{1}{2}p,\,\nu,\,\frac{1}{2}p}(t,u)du$$

$$-\frac{2^{2-p}(\nu+1)\alpha\,\Gamma(\frac{1}{2}p+\frac{1}{2}\nu+1)}{(\nu+2)\Gamma(\frac{1}{2}p+1)\Gamma(p+\frac{1}{2}\nu)}\int_c^1 u f_2(u)du \tag{5.27}$$

and the kernel is given by the equation

$$L(t,u) = \frac{2^{1-\frac{1}{2}p}\,t^{1-\nu-\frac{1}{2}p}\,u^{\nu+\frac{1}{2}p}}{\pi\Gamma(1+\frac{1}{2}p)}\sin(\tfrac{1}{2}p\pi)K_{\nu+1,\,\nu+\frac{1}{2}p,\,\nu+\frac{1}{2}p,\,0}(t,u)$$

$$+\frac{2^{2-p}\,\Gamma(\nu+1)\Gamma(\frac{1}{2}p+\frac{1}{2}\nu+1)t^{p-\nu-1}\,u^{2\nu+p}\alpha}{\Gamma(p+\frac{1}{2}\nu)\Gamma(\frac{1}{2}p+\nu+1)}. \tag{5.28}$$

In particular if we put $\nu = 0$, $p = 1$, $f_2(\rho) \equiv 0$ we see that the dual series equations

$$\alpha c_0 + \sum_{n=1}^{\infty} \lambda_n c_n J_0(\rho\lambda_n) = f(\rho), \quad 0 \leqslant \rho < c, \tag{5.29}$$

$$c_0 + \sum_{n=1}^{\infty} c_n J_0(\rho\lambda_n) = 0, \qquad c < \rho \leqslant 1, \tag{5.30}$$

where the $\{\lambda_n\}$ are the positive zeros of $J_1(\lambda)$, have solution

$$c_0 = 2\int_0^c u g(u)du, \qquad c_n = \frac{2}{\lambda_n J_0^2(\lambda_n)}\int_0^c g(t)\sin(t\lambda_n)dt, \tag{5.31}$$

where $g(t)$ is the solution of the Fredholm equation (5.26) with free term

$$\chi(t) = \frac{2}{\pi}\int_0^t \frac{v f(v)dv}{\sqrt{(t^2-v^2)}} \tag{5.32}$$

and kernel

$$L(t,u) = 2\alpha u + \left(\frac{2}{\pi}\right)^{5/2}\int_0^{\infty}\frac{K_1(y)}{I_1(y)}\sinh(ty)\sinh(uy)dy. \tag{5.33}$$

Solution in the case $p = -1$, $\nu = H = 0$.

It is difficult to derive the solution in the general case when $H + \nu = 0$ and $-1 \leqslant p < 0$; we shall consider only the two cases which are of most interest physically. The first problem we shall consider is the solution of the dual equations

$$\alpha c_0 + \sum_{n=1}^{\infty} \lambda_n^{-1} c_n J_0(\rho \lambda_n) = f(\rho), \quad 0 \leqslant \rho < c \tag{5.34}$$

$$c_0 + \sum_{n=1}^{\infty} c_n J_0(\rho \lambda_n) = 0, \qquad c < \rho \leqslant 1 \tag{5.35}$$

in which the function $f(\rho)$ is prescribed in $(0, c)$, α is a given constant and $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the positive

$$J_1(\lambda) = 0.$$

We suppose that when $0 \leqslant \rho < c$

$$c_0 + \sum_{n=1}^{\infty} c_n J_0(\rho \lambda_n) = -\frac{1}{\rho} \frac{\partial}{\partial \rho} \int_\rho^c \frac{t g(t) dt}{\sqrt{(t^2 - \rho^2)}} \tag{5.36}$$

then

$$c_0 = 2 \int_0^c g(t) dt, \qquad c_n = \frac{2}{J_0^2(\lambda_n)} \int_0^c g(t) \cos(t \lambda_n) dt. \tag{5.37}$$

Substituting these expressions for the coefficients c_n into the left-hand side of equation (5.34) we find that

$$2\alpha \int_0^c g(t) dt + \int_0^c g(t) dt \left(2 \sum_{n=1}^{\infty} \frac{J_0(\rho\lambda_n) \cos(t\lambda_n)}{\lambda_n J_0^2(\lambda_n)} \right) = f(\rho), \quad 0 \leqslant \rho < c, \tag{5.38}$$

However it can be shown that

$$2 \sum_{n=1}^{\infty} \frac{J_0(\rho\lambda_n) \cos(t\lambda_n)}{\lambda_n J_0^2(\lambda_n)} = (\rho^2 - t^2)^{-\frac{1}{2}} H(\rho - t) - 2(1 - t^2)^{\frac{1}{2}}$$

$$- \frac{2}{\pi} \int_0^\infty \frac{K_1(y)}{y I_1(y)} \cosh ty \left\{ 2 I_1(y) - y I_0(\rho y) \right\} dy \tag{5.39}$$

so that the equation (5.38) is equivalent to the relation

$$\int_0^{\rho} \frac{g(t)\,dt}{\sqrt{(\rho^2 - t^2)}} = \int_0^{c} g(u)\, H(u, \rho)\,du + f(\rho), \quad 0 \leq \rho < c \tag{5.40}$$

where the function $H(u, \rho)$ is defined by the equation

$$H(u, \rho) = 2\sqrt{(1 - u^2)} - 2\alpha + \frac{2}{\pi}\int_0^{\infty} \frac{K_1(y)}{yI_1(y)} \cosh(uy)\left\{2I_1(y) - yI_0(\rho y)\right\} dy. \tag{5.41}$$

Solving the equation (5.40) for $g(t)$ as though the right-hand side were a known function of ρ we obtain the integral equation

$$g(t) - \int_0^{c} g(u) K_1(u, t)\,du = \chi(t), \tag{5.42}$$

where

$$\chi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^{t} \frac{u f(u)\,du}{\sqrt{(t^2 - u^2)}} \tag{5.43}$$

and

$$K_1(u, t) = \frac{2}{\pi}\frac{d}{dt}\int_0^{t} \frac{\rho H(u, \rho)\,d\rho}{\sqrt{(t^2 - \rho^2)}}$$

$$= \frac{4}{\pi}\left\{\sqrt{(1 - u^2)} - \alpha\right\} + \frac{4}{\pi^2}\int_0^{\infty} \frac{K_1(y)}{yI_1(y)} \cosh(uy)\left\{2I_1(y) - y\cosh(ty)\right\} dy.$$

Using the integrals

$$\int_0^{\infty} \frac{K_1(y)}{y} (\cosh \alpha y - 1)\,dy = \tfrac{1}{2}\pi\left\{1 - \sqrt{(1 - \alpha^2)}\right\}$$

we find that

$$K_1(u, t) = \frac{4}{\pi}(1 - \alpha) + \frac{4}{\pi^2}\int_0^{\infty} \frac{K_1(y)}{yI_1(y)} \left\{2I_1(y) - y\cosh uy \cosh ty\right\} dy. \tag{5.44}$$

<u>Solution in the case $p = -1$, $\nu = -H = 1$.</u>

The second problem we shall discuss here is that of solving the dual series relations

$$\alpha c_0 \rho + \sum_{n=1}^{\infty} \lambda_n^{-1} c_n J_1(\rho\lambda_n) = f(\rho), \quad 0 \leq \rho < c, \tag{5.45}$$

$$c_0\rho + \sum_{n=1}^{\infty} c_n J_1(\rho\lambda_n) = 0, \qquad c < \rho \leqslant 1 \tag{5.46}$$

where, as before, the constant α and the function $f(\rho)$ are prescribed, but now $\lambda_1, \lambda_2, \ldots$ are the positive roots of the equation

$$J_2(\lambda) = 0.$$

We make the assumption that

$$c_0\rho + \sum_{n=1}^{\infty} c_n J_1(\rho\lambda_n) = -\frac{\partial}{\partial\rho}\int_\rho^c \frac{g(t)\,dt}{\surd(t^2-\rho^2)}, \quad 0 \leqslant \rho < c \tag{5.47}$$

from which it follows that

$$c_0 = 8\int_0^c ug(u)du, \qquad c_n = \frac{2}{J_1^2(\lambda_n)}\int_0^c g(u)\sin(\lambda_n u)du. \tag{5.48}$$

Substituting these values for the coefficients a_n in the left-hand side of equation (5.33) we obtain the relation

$$8\alpha\rho\int_0^c ug(u)du + \int_0^c g(t)dt\left\{2\sum_{n=1}^{\infty}\frac{J_1(\rho\lambda_n)\sin(t\lambda_n)}{\lambda_n J_1^2(\lambda_n)}\right\} = f(\rho), \; 0 \leqslant \rho < c. \tag{5.49}$$

Considering the Dini expansion of the function (of ρ) defined by the right hand side of equation (5.50) below it may be established that

$$2\sum_{n=1}^{\infty}\frac{J_1(\rho\lambda_n)\sin(t\lambda_n)}{\lambda_n J_1^2(\lambda_n)} = \frac{tH(\rho-t)}{\rho\surd(\rho^2-t^2)} - 4t\rho\surd(1-t^2)$$

$$-\frac{2}{\pi}\int_0^\infty \frac{K_2(y)}{yI_2(y)}\sinh(ty)\left\{4\rho I_2(y) - yI_1(\rho y)\right\}dy. \tag{5.50}$$

Inserting this expression for the infinite series occurring in the left-hand side of equation (5.37) we see that this latter equation reduces to

$$\int_0^\rho \frac{tg(t)dt}{/(\rho^2-t^2)} = \rho f(\rho) + \int_0^c g(u)\,H(u,\rho)du$$

where

$$H(u,\rho) = 4u\rho^2\surd(1-u^2) - 8\alpha\rho^2 u + \frac{2}{\pi}\int_0^\infty \frac{K_2(y)}{yI_2(y)}\sinh(uy)\left\{4\rho^2 I_2(y) - \rho y I_1(\rho y)\right\}dy.$$

This in turn leads to the Fredholm integral equation

$$tg(t) - \int_0^c K(u, t)du = \chi(t) \tag{5.51}$$

where

$$\chi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho^2 f(\rho)d\rho}{\sqrt{(t^2 - \rho^2)}} \tag{5.52}$$

and

$$K(u, t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\rho H(u,\rho)d\rho}{\sqrt{(t^2 - \rho^2)}}$$

$$= \frac{16}{\pi}ut^2\sqrt{(1-u^2)} - \frac{32}{\pi}\alpha u t^2$$

$$+ \frac{4}{\pi^2}\int_0^\infty \frac{K_2(y)}{yI_2(y)}\left\{8t^2 I_2(y) - ty\sinh(ty)\right\}\sinh(uy)dy.$$

Using the fact that

$$\frac{2}{\pi}\int_0^\infty K_2(y)\left(\frac{\sinh\theta y}{y} - \theta\right)dy = \theta - \theta\sqrt{(1-\theta^2)}$$

we find that

$$K(u, t) = \frac{16}{\pi}ut^2(1 - 2\alpha) + \frac{4}{\pi^2}\int_0^\infty \frac{K(y)}{yI_2(y)}\left\{8ut^2I_2(y) - ty\sinh(ty)\sinh(uy)\right\}dy \tag{5.53}$$

6. Dual Relations involving Sin Series.

We saw in §2 (Cf. equations (2.50) and 2.51) that certain mixed boundary value problems concerning plane harmonic functions lead to the solution of dual series relations of the type

$$\sum_{n=1}^\infty n^p a_n \sin nx = f(x), \quad 0 \leq x < c \tag{6.1}$$

$$\sum_{n=1}^\infty a_n \sin nx = 0, \qquad c < x \leq \pi. \tag{6.2}$$

In this section we shall consider only the cases $p = \pm 1$ since these are the ones which occur most frequently in applications.

6.1. The case in which $p = -1$.

If we assume that when $0 \leqslant x < c$

$$\sum_{n=1}^{\infty} a_n \sin nx = g(x) \tag{6.3}$$

then the constants a_n whose values we wish to determine are the Fourier sine coefficients of the function defined by the right-hand sides of equations (6.2) and (6.3). Hence we have

$$a_n = \frac{2}{\pi} \int_0^c g(t) \sin(nt)dt. \tag{6.4}$$

The problem of determining the a_n (or the $g(t)$ seems to have been considered first by Tranter (1959b) who made the integral representation

$$g(t) = \xi(1) \sin(\tfrac{1}{2}t)\,(1 - \sin^2 \tfrac{1}{2}t \operatorname{cosec}^2 \tfrac{1}{2}c)^{-\frac{1}{2}} - \int_T^1 \frac{\xi'(s)\sin(\frac{1}{2}x)ds}{\sqrt{(s^2 - T^2)}} \tag{6.5}$$

where T, the lower limit in the integral on the right is an abbreviation for $\sin\frac{1}{2}t \operatorname{cosec}\frac{1}{2}c$. Substituting this expression for $g(t)$ into equation (6.4) and then substituting the resulting expression for a_n into equation (6.1) it is possible to find an integral equation for $\xi(s)$ whose solution enables us to determine the function $g(t)$ and the set of constants $\left\{a_n\right\}$. Tranter's analysis is very complicated and for that reason we shall not repeat it here. It turns out in the end that

$$\xi(s) = \frac{1 - s^2 \sin^2 \frac{1}{2}c}{\pi s.\sin^2 \frac{1}{2}c} \cdot \frac{d}{ds} \int_0^s \frac{\rho f\, 2\left\{\sin^{-1}(\rho \sin\frac{1}{2}c)\right\} d\rho}{\sqrt{\left\{(s^2 - \rho^2)(1 - \rho^2 \sin \frac{1}{2}c)\right\}}}. \tag{6.6}$$

The function $\xi(s)$ is determined in terms of the known function $f(x)$ by means of this equation; $g(t)$ is then found from equation (6.5) and the constants a_n from (6.4).

Recently Williams (1963) has given a much more direct and simple method of solving the pair of equations (6.1) and (6.2). Substituting from equation (6.4) into equation (6.1) we find that the unknown function $g(t)$ is the solution of the integral equation

$$\int_0^c g(t)\,K(t,\,x)dt = \pi f(x), \quad 0 \leqslant x < c \tag{6.7}$$

in which the kernel $K(t, x)$ is defined by the equation

$$K(t,\,x) = 2\sum_{n=1}^{\infty} n^{-1}\sin(nx)\sin(nt)$$

$$= \log\left|\frac{\sin\frac{1}{2}(x+t)}{\sin\frac{1}{2}(x-t)}\right|. \tag{6.8}$$

By an ingenious method Williams reduces this kernel to a form which at first sight seems to make the integral equation (6.7) more complicated but which, in fact, enables us to solve it in explicit form. We consider the integral

$$I = \int_0^{\min(t,\,x)} \frac{\tan(\frac{1}{2}u)du}{\sqrt{\{(\cos u - \cos t)(\cos u - \cos x)\}}}.$$

The result will be symmetrical in $\underline{x}$ and $\underline{t}$ so for convenience we assume that $x > t$ and consider

$$I = \int_0^t \frac{\tan\frac{1}{2}u\,du}{\sqrt{\{(\cos u - \cos t)(\cos u - \cos x)\}}}.$$

If, in this integral, we change the variable of integration from u to v where

$$v^2 = \tan^2\tfrac{1}{2}t - \tan^2\tfrac{1}{2}u$$

we find that

$$I = \sec(\tfrac{1}{2}x)\sec(\tfrac{1}{2}t)\int_0^{\tan\frac{1}{2}t} \frac{dv}{\sqrt{(v^2 + \tan^2\frac{1}{2}x - \tan^2\frac{1}{2}t)}}$$

This integration is elementary and leads to the result

$$I = \tfrac{1}{2}\sec(\tfrac{1}{2}x)\sec(\tfrac{1}{2}t)\log\left|\frac{\tan\frac{1}{2}x + \tan\frac{1}{2}t}{\tan\frac{1}{2}x - \tan\frac{1}{2}t}\right|$$

$$= \tfrac{1}{2}\sec(\tfrac{1}{2}x)\sec(\tfrac{1}{2}t)\,K(t,\,x).$$

In other words

$$K(t, x) = 2 \cos\tfrac{1}{2}x \cos\tfrac{1}{2}t \int_0^{\min(t,x)} \frac{\tan\frac{1}{2}u \, du}{\sqrt{\{(\cos u - \cos t)(\cos u - \cos x)\}}}. \qquad (6.9)$$

If we now substitute this expression into equation (6.7) and interchange the order of the integrations in the integral on the right hand side we find that equation (6.7) can be written in the form

$$\cos\tfrac{1}{2}x \int_0^x \frac{\tan(\frac{1}{2}u)du}{\sqrt{(\cos u - \cos x)}} \int_u^c \frac{\cos(\frac{1}{2}t)\, g(t)dt}{\sqrt{(\cos u - \cos t)}} = \tfrac{1}{2}\pi f(x), \quad 0 \leq x < c. \qquad (6.10)$$

(Cf. Fig. 5 below). Hence if we write

$$G(u) = \int_u^c \frac{\cos(\frac{1}{2}t)\, g(t)dt}{\sqrt{(\cos u - \cos t)}} \qquad (6.11)$$

equation (6.10) becomes

$$\int_0^x \frac{\tan(\frac{1}{2}u)\, G(u)du}{\sqrt{(\cos u - \cos x)}} = \tfrac{1}{2}\pi \sec(\tfrac{1}{2}x)\, f(x), \quad 0 \leq x < c \qquad (6.12)$$

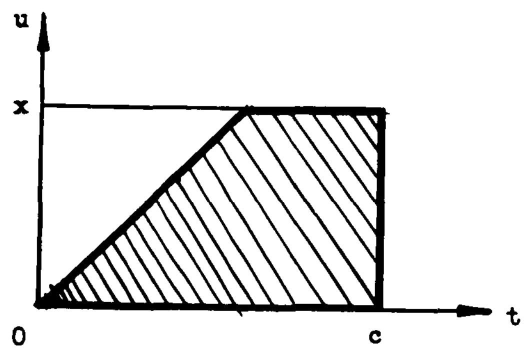

Fig. 5

Using equations (3.42 a and b) we see that the solution of equation (6.12) is

$$G(u) = \cot\tfrac{1}{2}u \frac{d}{du} \int_0^u \frac{\sin(\frac{1}{2}x)\, f(x)dx}{\sqrt{(\cos x - \cos u)}} \qquad (6.13)$$

and using equations (3.43 a and b) we see that the solution of equation (6.11) is

$$g(t) = \frac{-\sec\frac{1}{2}t}{\pi} \frac{d}{dt} \int_t^c \frac{\sin u\, G(u)du}{\sqrt{(\cos t - \cos u)}}. \qquad (6.14)$$

We can write this result in another way. Integrating with respect to t from x

to $\underline{c}$ we find that

$$\int_x^c g(t)dt = \frac{\sec\frac{1}{2}x}{\pi}\int_x^c \frac{\sin u\, G(u)du}{\surd(\cos x - \cos u)} + \frac{1}{2\pi}\int_x^c \sin u\, G(u)du \; \times$$

$$\times \int_x^u \frac{\sec\frac{1}{2}y \tan\frac{1}{2}y\, dy}{\surd(\cos y - \cos u)} . \tag{6.15}$$

The y-integral is easily reduced to the form

$$\frac{1}{\surd 2}\int_x^u \frac{\sec^2\frac{1}{2}y\tan\frac{1}{2}y\, dy}{\surd(1 - \cos^2\frac{1}{2}u\sec^2\frac{1}{2}y)} = \surd 2\Big[- \sec^2\tfrac{1}{2}u\surd(1-\cos^2\tfrac{1}{2}u\sec^2\tfrac{1}{2}y)\Big]_{y=x}^{y=u}$$

$$= \sec(\tfrac{1}{2}x)\sec^2(\tfrac{1}{2}u)\surd(\cos x - \cos u).$$

Inserting this expression into the double integral on the right-hand side of equation (6.15) we find that

$$\int_x^c g(t)dt = \frac{2}{\pi}\cos(\tfrac{1}{2}x)\int_x^c \frac{\tan(\frac{1}{2}u)\, G(u)}{\surd(\cos x - \cos u)}du .$$

Differentiating both sides of this equation with respect to $\underline{x}$ we find that

$$g(x) = -\frac{2}{\pi}\frac{d}{dx}\cos(\tfrac{1}{2}x)\int_x^c \frac{\tan(\frac{1}{2}u)\, G(u)}{\surd(\cos x - \cos u)}du, \quad 0 \leq x < c, \tag{6.16}$$

where $G(u)$ is defined by equation (6.13).

An alternative solution has been derived by Srivastav (1963) based on the integral representation

$$g(x) = -\frac{d}{dx}\cos\tfrac{1}{2}x\int_x^c \frac{\phi(t)dt}{\surd(\cos x - \cos t)}, \quad 0 < x < c \tag{6.17}$$

of the function $g(x)$ introduced in equation (6.3). Substituting this form into the right-hand side of equation (6.4) and integrating by parts we find that

$$a_n = \frac{2n}{\pi}\int_0^c \cos(nx)\cos(\tfrac{1}{2}x)dx\int_x^c \frac{\phi(t)dt}{\surd(\cos x - \cos t)} .$$

Interchanging the order of the integrations we find that

$$a_n = \frac{2n}{\pi}\int_0^c (t)dt\int_0^t \frac{\cos(\frac{1}{2}x)\cos(nx)}{\surd(\cos x - \cos t)}dx . \tag{6.18}$$

From the representations

$$P_{n-1}(\cos t) = \frac{\sqrt{2}}{\pi}\int_0^t \frac{\cos(n-\frac{1}{2})x\,dx}{\sqrt{(\cos x - \cos t)}} = \frac{\sqrt{2}}{\pi}\int_t^\pi \frac{\sin(n-\frac{1}{2})x\,dx}{\sqrt{(\cos t - \cos x)}} \tag{6.19}$$

of the Legendre polynomial (Whittaker and Watson, 1927, p.315) we immediately deduce that

$$\frac{\sqrt{2}}{\pi}\int_0^t \frac{\cos(\frac{1}{2}x)\cos(n x)}{\sqrt{(\cos x - \cos t)}}\,dx = \frac{\sqrt{2}}{\pi}\int_t^\pi \frac{\cos(\frac{1}{2}x)\sin(n x)}{\sqrt{(\cos x - \cos t)}} = \tfrac{1}{2}\left\{P_{n-1}(t) + P_n(t)\right\} \tag{6.20}$$

so that

$$a_n = \frac{n}{\sqrt{2}}\int_0^c \phi(t)\left\{P_{n-1}(t) + P_n(t)\right\}dt. \tag{6.21}$$

Substituting this expression for a_n into equation (6.1) we find that this equation is equivalent to

$$\int_0^c \phi(t)\,L(t, x)dt = f(x), \quad 0 < x < c \tag{6.22}$$

where the kernel $L(t, x)$ is defined by the equation

$$L(x, t) = \frac{1}{\sqrt{2}}\sum_{n=1}^{\infty}\left\{P_n(\cos t) + P_{n-1}(\cos t)\right\}\sin(nx).$$

Now it follows immediately from the second of the equations (6.20) that the series on the right hand side of this equation is the Fourier half-range sine series of the function

$$L(t, x) = \frac{\cos(\frac{1}{2}x)\,H(x-t)}{\sqrt{(\cos t - \cos x)}} \quad 0 \leqslant x \leqslant \pi.$$

Substituting this form for the kernel $L(t, x)$ into equation (6.22) we see that it is equivalent to the equation

$$\int_0^x \frac{\phi(t)\,dt}{\sqrt{(\cos t - \cos x)}} = \sec(\tfrac{1}{2}x)\,f(x), \quad 0 \leqslant x < c.$$

Using equations (3.42<u>a</u> and <u>b</u>) we can invert this equation to obtain the formula

$$\phi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\sin(\frac{1}{2}x)\,f(x)}{\sqrt{(\cos x - \cos t)}}\,dx \tag{6.23}$$

for the determination of the function $\phi(t)$. If we introduce the function G defined by equation (6.13) we find that

$$\phi(t) = \tan\tfrac{1}{2}t\,G(t). \tag{6.24}$$

Substituting this expression for $\phi(t)$ into the right hand side of equation (6.17) we obtain, once again, the equation (6.16) for the function $g(x)$ in the interval $(0, c)$.

6.2. The case in which $p = +1$.

To solve the equations

$$\sum_{n=1}^{\infty} n\, a_n \sin nx = f(x), \qquad 0 < x < c \tag{6.25}$$

$$\sum_{n=1}^{\infty} a_n \sin nx = 0, \qquad c < x < \pi \tag{6.26}$$

we integrate both sides of equation (6.25) with respect to x to obtain the equivalent relation

$$\sum_{n=1}^{\infty} a_n (1 - \cos nx) = \int_0^x f(u)du, \quad 0 < x < c. \tag{6.27}$$

If we now assume that, in the interval $0 < x < c$,

$$\sum_{n=1}^{\infty} a_n \sin nx = \sin(\tfrac{1}{2}x) \int_x^c \frac{g(t)dt}{\sqrt{(\cos x - \cos t)}}, \tag{6.28}$$

it follows from the theory of Fourier series that

$$a_n = \frac{2}{\pi}\int_0^c \sin(nx)\sin(\tfrac{1}{2}x)\,dx \int_x^c \frac{g(t)dt}{\sqrt{(\cos x - \cos t)}}$$

$$= \frac{1}{\pi}\int_0^c g(t)dt \int_0^t \frac{\cos(n-\frac{1}{2})x - \cos(n+\frac{1}{2})x}{\sqrt{(\cos x - \cos t)}}\,dx.$$

Making use of the integral representation (6.19) for the Legendre polynomial we see that this equation can be written in the form

$$a_n = \frac{1}{\sqrt{2}} \int_0^c g(t)\left[P_{n-1}(\cos t) - P_n(\cos t)\right]dt. \tag{6.29}$$

If we substitute this expression for the coefficients a_n into equation (6.27) and then interchange the order of summation and integration we obtain the integral equation

$$\int_0^c g(t)\, S(x,\ t)\,dt = \int_0^x f(u)du, \qquad 0 < x < c \tag{6.30}$$

where

$$S(x,\ t) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} (1 - \cos nx)\left\{P_{n-1}(\cos t) - P_n(\cos t)\right\}$$

$$= \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \left\{P_n(\cos t) - P_{n-1}(\cos t)\right\}\cos nx.$$

By considering the half-range Fourier cosine series of the function

$$\frac{\sin(\frac{1}{2}x)\, H(x - t)}{\sqrt{(\cos t - \cos x)}} \tag{6.31}$$

we can show that $S(x, t)$ is equal to this function. Substituting the expression (6.31) for $S(x, t)$ into the left-hand side of equation (6.30) we obtain the integral equation

$$\int_0^x \frac{g(t)\, dt}{\sqrt{(\cos t - \cos x)}} = \operatorname{cosec}(\tfrac{1}{2}x) \int_0^x f(u)du \tag{6.32}$$

for the determination of the function $g(t)$. The solution of this equation is given by equations (3.42<u>a</u> and <u>b</u>) in the form

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\cos(\frac{1}{2}x)}{\sqrt{(\cos x - \cos t)}}\, dx \int_0^x f(u)du. \tag{6.33}$$

6.3. Solution of Equations of the Second Type.

The solution of the equations

$$\sum_{n=1}^{\infty} n^p a_n \sin nx = 0, \qquad 0 \leqslant x < c \tag{6.34}$$

$$\sum_{n=1}^{\infty} a_n \sin nx = f(x), \quad c < x \leqslant \pi \tag{6.35}$$

can readily be deduced from that of equations (6.1) and (6.2). If we change the independent variable from x to $y = \pi - x$ and write $(-1)^{n+1} n^p a_n = A_n$, $\gamma = \pi - c$ we find that these equations are equivalent to

$$\sum_{n=1}^{\infty} n^{-p} A_n \sin ny = f_1(y), \qquad 0 \leqslant y < \gamma \tag{6.36}$$

$$\sum_{n=1}^{\infty} A_n \sin ny = 0, \qquad \gamma < y \leqslant \pi \tag{6.37}$$

where

$$f_1(y) = f(\pi - y). \tag{6.38}$$

If $p = 1$ the solution of equations (6.36) and (6.37) is given by equations (6.21) and (6.23) in the form

$$A_n = \frac{n}{\sqrt{2}} \int_0^{\gamma} \phi(t) \left\{ P_{n-1}(\cos t) + P_n(\cos t) \right\} dt \tag{6.39}$$

where

$$\phi(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{\sin(\frac{1}{2} y)\, f_1(y)}{\sqrt{(\cos y - \cos t)}}\, dy. \tag{6.40}$$

Reverting to the original variables we find that

$$a_n = \frac{1}{\sqrt{2}} \int_c^{\pi} \psi(t) \left\{ P_n(\cos t) - P_{n-1}(\cos t) \right\} dt \tag{6.41}$$

with

$$\psi(t) = \frac{2}{\pi} \frac{d}{dt} \int_t^{\pi} \frac{f(x) \cos(\frac{1}{2} x)}{\sqrt{(\cos t - \cos x)}}\, dx \tag{6.42}$$

is the solution of the pair of dual integral equations

$$\sum_{n=1}^{\infty} n\, a_n \sin nx = 0, \qquad 0 \leqslant x < c \tag{6.43}$$

$$\sum_{n=1}^{\infty} a_n \sin nx = f(x), \qquad c < x \leqslant \pi. \tag{6.44}$$

If $p = -1$ the solution of equations (6.36) and (6.37) is given by equations (6.29) and (6.33) in the form

$$A_n = \frac{1}{\sqrt{2}} \int_0^{\gamma} \phi(t) \left\{ P_{n-1}(\cos t) - P_n(\cos t) \right\} dt$$

where
$$\phi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\cos(\frac{1}{2}y)}{\sqrt{(\cos y - \cos t)}}\,dy\int_0^y f_1(u)du$$

Reverting to the original variables we find that

$$a_n = \frac{n}{\sqrt{2}}\int_c \psi(t)\left\{P_n(\cos t) + P_{n-1}(\cos t)\right\}dt \tag{6.45}$$

with
$$\psi(t) = -\frac{2}{\pi}\frac{d}{dt}\int_t^\pi \frac{\sin(\frac{1}{2}x)dx}{\sqrt{(\cos t - \cos x)}}\int_x^\pi f(u)du = -\frac{\sin(\frac{1}{2}t)}{\pi}\int_t^\pi \frac{\cos(\frac{1}{2}u)f'(u)du}{\sqrt{(\cos t - \cos u)}} \tag{6.46}$$

is the solution of the pair of dual integral equations

$$\sum_{n=1}^{\infty} n^{-1} a_n \sin nx = 0, \qquad 0 \leqslant x < c, \tag{6.47}$$

$$\sum_{n=1}^{\infty} a_n \sin nx = f(x) \qquad c < x \leqslant \pi. \tag{6.48}$$

6.4. Sine Series Analogous to Dini Series.

Finally we shall consider the solution of the pair of dual integral equations

$$\sum_{n=1}^{\infty} (n-\tfrac{1}{2})^p a_n \sin(n-\tfrac{1}{2})x = f_1(x), \quad 0 \leqslant x < c, \tag{6.49}$$

$$\sum_{n=1}^{\infty} a_n \sin(n-\tfrac{1}{2})x = f_2(x), \quad c < x \leqslant \pi. \tag{6.50}$$

where $p = \pm 1$, following the method of Srivastav (1963b).

Case (a): $p = 1$, $f_2(x) \equiv 0$. To find the solution in this case we assume that when $0 \leqslant x < c$

$$\sum_{n=1}^{\infty} a_n \sin(n-\tfrac{1}{2})x = \sin x\int_x^c \frac{g_1(t)\,dt}{\sqrt{(\cos x - \cos t)}}, \; 0 \leqslant x < c \tag{6.51}$$

from which it follows that

$$a_n = \frac{2}{\pi}\int_0^c \sin(n-\tfrac{1}{2})x \,.\, \sin x\, dx\int_x^c \frac{g_1(t)\,dt}{\sqrt{(\cos x - \cos t)}}$$

$$= \frac{1}{\pi} \int_0^c g_1(t)dt \int_0^t \frac{\cos(n - \frac{3}{2})x - \cos(n + \frac{1}{2})x}{\sqrt{(\cos x - \cos t)}} dx.$$

Using equation (6.19) we find that

$$\left.\begin{aligned} a_1 &= \frac{1}{\sqrt{2}} \int_0^c g_1(t)\left[1 - P_1(\cos t)\right]dt \\ a_n &= \frac{1}{\sqrt{2}} \int_0^c g_1(t)\left[P_{n-2}(\cos t) - P_n(\cos t)\right]dt, \quad n \geq 2. \end{aligned}\right\} \qquad (6.52)$$

If we integrate both sides of equation (6.49) from 0 to $x < c$ we get the relation

$$\sum_{n=1}^{\infty} a_n\left[1 - \cos(n - \tfrac{1}{2})x\right] = \int_0^x f_1(u)du. \qquad (6.53)$$

Substituting the expressions (6.52) for the coefficients a_n and interchanging the order of summation and integration we obtain the relation

$$\frac{1}{\sqrt{2}} \int_0^c g_1(t)\, S(t, x)dt = \int_0^x f_1(u)du \qquad (6.54)$$

where

$$S(t, x) = \left[1 - \cos(\tfrac{1}{2}x)\right]\left[1 - P_1(\cos t)\right] + \sum_{n=2}^{\infty} \left\{P_{n-2}(\cos t) - P_n(\cos t)\right\}\left\{1 - \cos(n - \tfrac{1}{2})x\right\}.$$

Now it is easily shown that

$$\frac{\sqrt{2}\sin x\, H(x - t)}{\sqrt{(\cos t - \cos x)}} = \left[1 + P_1(\cos t)\right]\cos(\tfrac{1}{2}x) + \sum_{n=2}^{\infty}\left\{P_n(\cos t) - P_{n-2}(\cos t)\right\}\cos(n - \tfrac{1}{2})x$$

and that

$$\sum_{n=2}^{\infty}\left\{P_{n-2}(\cos t) - P_n(\cos t)\right\} = 1 + P_1(\cos t)$$

from which it follows that

$$S(t, x) = 2\left[1 - \cos(\tfrac{1}{2}x)\right] + \frac{\sqrt{2}\sin x\, H(x - t)}{\sqrt{(\cos t - \cos x)}}.$$

If we substitute this expression for $S(t, x)$ into equation (6.54) we obtain the integral equation

$$\sin x \int_0^x \frac{g_1(t)\,dt}{\sqrt{(\cos t - \cos x)}} = \int_0^x f_1(u)du - \sqrt{2}\int_0^c (1 - \cos\tfrac{1}{2}x)g_1(t)dt$$

for the determination of $g_1(t)$. If we write

$$\gamma_1 = \int_0^c g_1(u)du \tag{6.55}$$

and solve this integral equation by making use of equation (3.42b) we find that

$$g_1(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t \frac{dx}{\sqrt{(\cos x - \cos t)}}\int_0^x f_1(u)du - \frac{\sqrt{2}\gamma_1}{\pi}\frac{d}{dt}\int_0^t \frac{1 - \cos\frac{1}{2}x}{\sqrt{(\cos x - \cos t)}}dx. \tag{6.56}$$

To obtain the value of the constant γ_1 we integrate both sides of this equation with respect to t from 0 to c. In this way we get the equation

$$\gamma_1\left\{1 + \frac{\sqrt{2}}{\pi}\int_0^c \frac{(1 - \cos\frac{1}{2}x)dx}{\sqrt{(\cos x - \cos c)}}\right\} = \frac{1}{\pi}\int_0^c \frac{dx}{\sqrt{(\cos x - \cos c)}}\int_0^x f_1(u)du \tag{6.57}$$

from which to calculate γ_1.

<u>Case (b): $p = 1,\ f_1(x) \equiv 0$.</u>

In a similar way we obtain the solution in this case by writing

$$\sum_{n=1}^{\infty}(n - \tfrac{1}{2})a_n \sin(n - \tfrac{1}{2})x = -\frac{d}{dx}\sin x\int_c^x \frac{g(t)\,dt}{\sqrt{(\cos t - \cos x)}},\quad c < x \leq \pi, \tag{6.58}$$

which is equivalent to assuming

$$\left.\begin{aligned} a_1 &= \frac{1}{\sqrt{2}}\int_c^{\pi} g_2(t)\left\{1 + P_1(\cos t)\right\}dt \\ a_n &= \frac{1}{\sqrt{2}}\int_c^{\pi} g_2(t)\left\{P_n(\cos t) - P_{n-2}(\cos t)\right\}dt. \end{aligned}\right\} \tag{6.59}$$

By a procedure similar to that outlined for case (a) we can show that

$$g_2(t) = \frac{1}{\pi}\frac{d}{dt}\int_t^{\pi} \frac{f_2(x)\,dx}{\sqrt{(\cos t - \cos x)}}. \tag{6.60}$$

<u>Case (c): $p = -1,\ f(x) \equiv 0$.</u>

To derive the solution in this case we begin with the assumption that when $0 \leq x < c$

$$\sum_{n=1}^{\infty} a_n \sin(n - \tfrac{1}{2})x = -\frac{d}{dx}\int_x^c \frac{g_1(t)\,dt}{\sqrt{(\cos x - \cos t)}} \tag{6.61}$$

which is the same as assuming that

$$a_n = \frac{n - \frac{1}{2}}{\sqrt{2}} \int_0^c g_1(t)\, P_{n-1}\,(\cos t)\,dt \tag{6.62}$$

$(n = 1, 2, \ldots)$. Substituting this expression for a_n into the series on the left-hand side of equation (6.49) with $p = -1$ and using a procedure similar to that adopted for case (a) we can show that

$$g_1(t) = \frac{1}{\pi}\frac{d}{dt}\int_0^t \frac{\sin x\, f_1(x)\,dx}{\sqrt{(\cos t - \cos x)}}. \tag{6.63}$$

Case (d): $p = -1,\ f_1(x) \equiv 0.$

To solve the equations

$$\sum_{n=1}^{\infty} (n - \tfrac{1}{2})^{-1}\, a_n \sin(n - \tfrac{1}{2})x = 0, \qquad 0 \leqslant x < c, \tag{6.64}$$

$$\sum_{n=1}^{\infty} a_n \sin(n - \tfrac{1}{2})x = f_2(x), \quad c < x \leqslant \pi. \tag{6.65}$$

we integrate equation (6.65) with respect to x from $x(> c)$ to π to obtain the equivalent relation

$$\sum_{n=1}^{\infty} (n - \tfrac{1}{2})^{-1}\, a_n \cos(n - \tfrac{1}{2})x = \int_x^{\pi} f_2(u)\,du. \tag{6.66}$$

We now set

$$\sum_{n=1}^{\infty} (n - \tfrac{1}{2})^{-1}\, a_n \sin(n - \tfrac{1}{2})x = \int_c^x \frac{g_2(t)\,dt}{\sqrt{(\cos t - \cos x)}}, \quad c < x < \pi \tag{6.67}$$

which is equivalent to the assumption

$$a_n = \frac{n - \frac{1}{2}}{\sqrt{2}} \int_c^{\pi} g_2(t)\, P_{n-1}(\cos t)\,dt. \tag{6.68}$$

Inserting this expression for the coefficient a_n in the series on the left hand side of equation (6.66), interchanging the order of integration and summation and making use of the result

$$\frac{H(t-x)}{\sqrt{(\cos x - \cos t)}} = \sum_{n=1}^{\infty} P_{n-1}(\cos t)\cos(n-\tfrac{1}{2})x \tag{6.69}$$

we find that

$$g_2(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^{\pi}\frac{\sin x\, dx}{\sqrt{(\cos t - \cos x)}}\int_x^{\pi} f_2(u)du. \tag{6.70}$$

After a little manipulation it can be shown that

$$g_2(t) = -\frac{2\sin t}{\pi}\int_t^{\pi}\frac{f_2(u)\, du}{\sqrt{(\cos t - \cos u)}}\,. \tag{6.71}$$

7. Dual Relations involving Cosine Series.

In this section we shall consider dual series relations of the type

$$\tfrac{1}{2}\lambda a_0 + \sum_{n=1}^{\infty} n^p a_n \cos nx = f_1(x), \qquad 0 \leqslant x < c$$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx = f_2(x), \qquad c < x \leqslant \pi$$

and other dual equations involving cosine series. We saw in §2.3 (see equations (2.45) and (2.46) above) that equations of this type with the value of λ prescribed arise in the analysis of certain physical problems. In what follows we shall always assume that the value of the constant λ is among the physical data of the problem. We shall follow the treatment given in Srivastav (1963b).

7.1. The case in which $p = +1$.

We begin by considering the pair of equations

$$\tfrac{1}{2}\lambda a_0 + \sum_{n=1}^{\infty} n\, a_n \cos(nx) = f(x), \qquad 0 < x < c \tag{7.1}$$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = 0, \qquad c < x < \pi\,. \tag{7.2}$$

Integrating the first of these two equations with respect to x from 0 to x we obtain the relation

$$\tfrac{1}{2}\lambda a_0 x + \sum_{n=1}^{\infty} a_n \sin(nx) = F(x), \quad 0 < x < c \tag{7.3}$$

where $F(x) = \int_0^x f(t)\,dt$.

If we assume that for $0 < x < c$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = \int_x^c \frac{g(u)\,du}{\sqrt{(u^2 - x^2)}} \tag{7.4}$$

then, from the formulae for determining the Fourier coefficients of a half-range cosine series, we have the equations

$$a_0 = \frac{2}{\pi}\int_0^c dx \int_x^c \frac{g(u)\,du}{\sqrt{(u^2 - x^2)}}, \quad a_n = \frac{2}{\pi}\int_0^c \cos(nx)\,dx \int_x^c \frac{g(u)\,du}{\sqrt{(u^2 - x^2)}}$$

Interchanging the orders of the integrations in these double integrals we obtain the expressions

$$a_0 = \int_0^c g(u)\,du, \qquad a_n = \int_0^c g(u)\, J_0(nu)\,du \tag{7.5}$$

If we substitute these expressions into equation (7.3) and interchange the order of the summation and integration, we obtain the relation

$$\int_0^c g(u)\left\{\sum_{n=1}^{\infty} J_0(nu)\sin(nx)\right\} du = F(x) - \tfrac{1}{2}\lambda x \int_0^c g(u)\,du, \quad 0 < x < c. \tag{7.6}$$

Using the result (3.36) we see that this equation is equivalent

$$\int_0^x \frac{g(u)\,du}{\sqrt{(x^2 - u^2)}} = F(x) - \tfrac{1}{2}\lambda x \int_0^c g(u)\,du + \int_0^c g(u)\,du \int_0^\infty \frac{\sinh(xy)}{\sinh(\pi y)} \times$$

$$\times\, e^{-\pi y} I_0(uy)\,dy, \qquad 0 < x < c.$$

If we regard this as an integral of equation of Abel type whose right-hand side is a known function of x we obtain the Fredholm equation

$$g(u) - \int_0^c g(t)\, K(t, u)\,dt = h(u), \qquad 0 < u < c. \tag{7.7}$$

where $h(u)$ denotes the free term

$$h(u) = \frac{2}{\pi}\frac{\partial}{\partial u}\int_0^u \frac{x F(x)\,dx}{\sqrt{(u^2 - x^2)}} = \frac{2}{\pi}\int_0^u \frac{f(x)\,dx}{\sqrt{(u^2 - x^2)}} \tag{7.8}$$

and $K(t, u)$ denotes the kernel

$$K(t, u) = u\int_0^\infty \frac{y e^{-\pi y}}{\sinh(\pi y)} I_0(ty)\, I_0(uy)\,dy - \tfrac{1}{2}\lambda u. \tag{7.9}$$

An alternative solution leading to an integral equation whose solution can be reduced to simple quadratures has also been derived by Srivastav. We put

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \cos(\tfrac{1}{2}x)\int_c^x \frac{\phi(t)\,dt}{\sqrt{(\cos x - \cos t)}}, \quad 0 \le x < c \tag{7.10}$$

from which it follows that

$$a_0 = \frac{1}{\sqrt{2}}\int_0^c \phi(t)\,dt, \quad a_n = \frac{1}{\sqrt{2}}\int_0^c \phi(t)\left\{P_n(\cos t) + P_{n-1}(\cos t)\right\}dt. \tag{7.11}$$

Substituting these expressions in equation (7.3) we obtain the relation

$$\frac{\lambda}{2\sqrt{2}}\, x\int_0^c \phi(t)\,dt + \frac{1}{\sqrt{2}}\int_0^c \phi(t)\left[\sum_{n=1}^{\infty}\left\{P_n(\cos t) + P_{n-1}(\cos t)\right\}\sin(nx)\right]dt = F(x),\ 0 \le x < c$$

from which, as in the derivation of equation (6.22) we deduce that

$$\frac{\lambda}{2\sqrt{2}}\, x\int_0^c \phi(t)\,dt + \cos(\tfrac{1}{2}x)\int_0^x \frac{\phi(t)\,dt}{\sqrt{(\cos t - \cos x)}} = F(x), \quad 0 \le x < c.$$

Inverting this equation by means of equations (3.42 a and b) we find that

$$\phi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\sin(\tfrac{1}{2}x)\, F(x)\,dx}{\sqrt{(\cos x - \cos t)}} - \frac{1}{\pi}\lambda a_0 \frac{d}{dt}\int_0^t \frac{x\sin(\tfrac{1}{2}x)\,dx}{\sqrt{(\cos x - \cos t)}}.$$

To determine a_0 we integrate both sides of this equation with respect to t from 0 to c to obtain the equation

$$a_0 = \frac{I_1}{1 + \lambda I_2} \tag{7.12}$$

where

$$I_1 = \frac{\sqrt{2}}{\pi}\int_0^c \frac{\sin(\tfrac{1}{2}x)\, F(x)\,dx}{\sqrt{(\cos x - \cos c)}}, \quad I_2 = \frac{1}{\pi\sqrt{2}}\int_0^c \frac{x\sin(\tfrac{1}{2}x)\,dx}{\sqrt{(\cos x - \cos c)}}. \tag{7.13}$$

Hence we have

$$\phi(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\sin(\frac{1}{2}x)\,F(x)\,dx}{\sqrt{(\cos x - \cos t)}} - \frac{\lambda I_1}{\pi(1+\lambda I_2)}\frac{d}{dt}\int_0^t \frac{x\sin(\frac{1}{2}x)\,dx}{\sqrt{(\cos x - \cos t)}}. \tag{7.14}$$

7.2. The case in which p = -1.

We now consider the solution of the pair of dual series relations

$$\tfrac{1}{2}\lambda a_0 + \sum_{n=1}^{\infty} n^{-1} a_n \cos(nx) = f(x), \qquad 0 \leqslant x < c \tag{7.15}$$

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = 0, \qquad c < x \leqslant \pi. \tag{7.16}$$

If we integrate the second of these series term by term we find that

$$\tfrac{1}{2}a_0(\pi - x) - \sum_{n=1}^{\infty} n^{-1}a_n \sin nx = 0, \qquad c < x \leqslant \pi. \tag{7.17}$$

We now assume that when $c < x \leqslant \pi$

$$\tfrac{1}{2}\lambda a_0 + \sum_{n=1}^{\infty} n^{-1} a_n \cos(nx) = \sin(\tfrac{1}{2}x)\int_c^x \frac{g(t)\,dt}{\sqrt{(\cos t - \cos x)}} \tag{7.18}$$

which is equivalent to assuming that

$$a_0 = \frac{2}{\pi}\left\{\int_0^c f(u)du + \frac{1}{\sqrt{2}}\int_c^\pi g(t)dt\right\} \tag{7.19}$$

$$a_n = n\,b_n + \frac{n}{\sqrt{2}}\int_c^\pi g(t)\Big[P_n(\cos t) - P_{n-1}(\cos t)\Big]dt \tag{7.20}$$

where

$$b_n = \frac{2}{\pi}\int_0^c f(u)\cos(nu)\,du. \tag{7.21}$$

If we substitute these values into equation (7.17) and interchange the order of integration and summation we find that

$$\int_c^\pi g(t)S(x,t)dt = \tfrac{1}{2}a_0(\pi - x) - \sum_{n=1}^{\infty} b_n \sin(nx), \qquad c < x \leqslant \pi$$

where

$$S(x, t) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \left[P_n(\cos t) - P_{n-1}(\cos t) \right] \sin(nx)$$

$$= \frac{\sin(\frac{1}{2} x)\, H(t - x)}{\sqrt{(\cos x - \cos t)}}$$

so that $g(t)$ is the solution of the integral equation

$$\sin(\tfrac{1}{2} x) \int_x^{\pi} \frac{g(t)\,dt}{\sqrt{(\cos x - \cos t)}} = \sum_{n=1}^{\infty} b_n \sin nx - \tfrac{1}{2} a_0(\pi - x) \qquad (7.22)$$

with solution

$$g(t) = \frac{1}{\sqrt{2}} \frac{d}{dt} \sum_{n=1}^{\infty} b_n \left\{ P_n(\cos t) + P_{n-1}(\cos t) \right\} - \frac{a_0}{\pi} \frac{d}{dt} \int_t^{\pi} \frac{(\pi - x)\cos(\frac{1}{2}x)\,dx}{\sqrt{(\cos t - \cos x)}} .$$

Now

$$\sum_{n=1}^{\infty} b_n \left\{ P_n(\cos t) + P_{n-1}(\cos t) \right\} = \frac{2}{\pi} \int_0^c f(u)\, \sigma(u, t)\, du$$

where

$$\sigma(u, t) = \sum_{n=1}^{\infty} \left\{ P_n(\cos t) + P_{n-1}(\cos t) \right\} \cos(nu), \qquad (0 \leq u \leq c \leq t \leq \pi)$$

$$= \frac{\sqrt{2} \cos(\frac{1}{2} u)\, H(t - u)}{\sqrt{(\cos u - \cos t)}} - 1$$

and hence

$$\sum_{n=1}^{\infty} b_n \left\{ P_n(\cos t) + P_{n-1}(\cos t) \right\} = \frac{2\sqrt{2}}{\pi} \int_0^c \frac{f(u) \cos(\frac{1}{2} u)\, du}{\sqrt{(\cos u - \cos t)}} - \frac{2}{\pi} \int_0^c f(u)\, du$$

showing that

$$\frac{1}{\sqrt{2}} \frac{d}{dt} \sum_{n=1}^{\infty} b_n \left\{ P_n(\cos t) + P_{n-1}(\cos t) \right\} = \frac{2}{\pi} \frac{d}{dt} \int_0^c \frac{f(u) \cos(\frac{1}{2} u)\, du}{\sqrt{(\cos u - \cos t)}} .$$

The solution of the integral equation (7.22) can therefore be written in the form

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^c \frac{f(u)\cos(\frac{1}{2}u)\,du}{\sqrt{(\cos u - \cos t)}} - \frac{a_0}{\pi}\frac{d}{dt}\int_t^\pi \frac{(\pi - x)\cos(\frac{1}{2}x)\,dx}{\sqrt{(\cos t - \cos x)}}. \tag{7.23}$$

The value of a_0 can be found by a simple integration.

7.3. Equations of the Second Type.

If we make the substitutions

$$c = \pi - \gamma;\quad x = \pi - y,\ \lambda a_0 = A_0;\ \lambda^{-1} = \Lambda;\ n(-1)^n a_n = A_n,\ f(\pi - y) = f_1(y)$$

in the equations

$$\tfrac{1}{2}\lambda a_0 + \sum_{n=1}^{\infty} n\, a_n \cos(nx) = 0, \qquad 0 \leqslant x < c \tag{7.24}$$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = f(x), \qquad 0 < x \leqslant \pi \tag{7.25}$$

we find that they take the form

$$\tfrac{1}{2}\Lambda A_0 + \sum_{n=1}^{\infty} n^{-1} A_n \cos(ny) = f_1(y), \qquad 0 \leqslant y < \gamma$$

$$\tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos(ny) = 0, \qquad \gamma < y \leqslant \pi$$

which have been considered in §7.2 above. From equations (7.19), (7.20) and (7.23) we can write down the solution of these equations and reverting to the original variables we find that the solution of equations (7.24) and (7.25) can be written in the form

$$\left.\begin{aligned} \lambda a_0 &= \frac{2}{\pi}\left\{\frac{1}{\sqrt{2}}\int_0^c g(t)\,dt + \int_c^\pi f(x)\,dx\right\} \\ a_n &= \frac{2}{\pi}\int_c^\pi f(x)\cos(nx)\,dx + \frac{1}{\sqrt{2}}\int_0^c \times \\ &\quad \times \left\{P_n(\cos t) + P_{n-1}(\cos t)\right\} g(t)\,dt \end{aligned}\right\} \tag{7.26}$$

where $g(t)$ is given by the equation

$$g(t) = -\frac{2}{\pi}\frac{d}{dt}\int_c^{\pi}\frac{f(u)\sin(\frac{1}{2}u)\,du}{\sqrt{(\cos t - \cos u)}} - \frac{\lambda a_0}{\pi}\frac{d}{dt}\int_0^{t}\frac{x\sin(\frac{1}{2}x)\,dx}{\sqrt{(\cos x - \cos t)}} \tag{7.27}$$

In a similar way we can reduce the solution of the pair of dual series equations

$$\tfrac{1}{2}\lambda a_0 + \sum_{n=1}^{\infty} n^{-1} a_n \cos nx = 0, \qquad 0 \leq x < c \tag{7.28}$$

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx = f(x), \qquad c < x \leq \pi \tag{7.29}$$

can be reduced to the type considered in §7.1 above. In this way we find that the solution of this pair of equations can be written in the form

$$a_0 = \frac{\sqrt{2}}{\lambda}\int_c^{\pi} g(t)\,dt \tag{7.30}$$

$$a_n = \frac{n}{\sqrt{2}}\int_c^{\pi} g(t)\left\{P_n(\cos t) - P_{n-1}(\cos t)\right\}dt \tag{7.31}$$

where

$$g(t) = -\frac{2}{\pi}\frac{d}{dt}\int_t^{\pi} f(u)\,du\int_t^{u}\frac{\cos(\frac{1}{2}x)\,dx}{\sqrt{(\cos t - \cos x)}} - a_0\frac{d}{dt}\int_t^{\pi}\frac{(\pi - x)\cos(\frac{1}{2}x)}{\sqrt{(\cos t - \cos x)}}\,dx. \tag{7.32}$$

7.4. Cosine Series analogous to Fourier-Bessel Series.

We now consider dual series equations of the type

$$\sum_{n=1}^{\infty}(n - \tfrac{1}{2})^p a_n \cos(n - \tfrac{1}{2})x = f_1(x), \qquad 0 \leq x < c \tag{7.33}$$

$$\sum_{n=1}^{\infty} a_n \cos(n - \tfrac{1}{2})x = f_2(x), \qquad c < x \leq \pi. \tag{7.34}$$

If we make the substitutions $x = \pi - y$, $c = \pi - \gamma$, $f_1(\pi - y) = F_1(y)$, $f_2(\pi - y) = F_2(y)$, $a_n = (-1)^n (n - \frac{1}{2})^{-p} A$ we find that these equations assume the forms

$$\sum_{n=1}^{\infty} (n - \tfrac{1}{2})^{-p} A_n \sin(n - \tfrac{1}{2})y = F_2(y), \qquad 0 \leq y < \gamma \tag{7.35}$$

$$\sum_{n=1}^{\infty} A_n \sin(n - \tfrac{1}{2})y = F_1(y), \qquad \gamma < y \leq \pi \tag{7.36}$$

and these equations have already been considered in §6.4.

In this way we can show that the equations

$$\sum_{n=1}^{\infty} (n - \tfrac{1}{2})\, a_n \cos(n - \tfrac{1}{2})x = f(x), \qquad 0 \leq x < c \tag{7.37}$$

$$\sum_{n=1}^{\infty} a_n \cos(n - \tfrac{1}{2})\, x = 0, \qquad c < x \leq \pi \tag{7.38}$$

have the solution

$$a_n = \sqrt{2} \int_0^c g(t)\, P_{n-1}(\cos t)\, dt \tag{7.39}$$

where

$$g(t) = \frac{\sin t}{\pi} \int_c^t \frac{f(u)\, du}{\sqrt{(\cos u - \cos t)}}. \tag{7.40}$$

This solution also has the property that when $0 \leq x < c$

$$\sum_{n=1}^{\infty} a_n \cos(n - \tfrac{1}{2})x = \int_x^c \frac{g(t)\, dt}{\sqrt{(\cos x - \cos t)}}. \tag{7.41}$$

The equations

$$\sum_{n=1}^{\infty} (n - \tfrac{1}{2})\, a_n \cos(n - \tfrac{1}{2})\, x = 0, \qquad 0 \leq x < c \tag{7.42}$$

$$\sum_{n=1}^{\infty} a_n \cos(n - \tfrac{1}{2})\, x = f(x), \quad c < x \leq \pi \tag{7.43}$$

have the solution

$$a_n = \sqrt{2}\int_c^{\pi} g(t)\, P_{n-1}(\cos t)\, dt \tag{7.44}$$

where

$$g(t) = -\frac{1}{\pi}\frac{d}{dt}\int_t^{\pi}\frac{f(x)\sin x\, dx}{\sqrt{(\cos t - \cos x)}} \tag{7.45}$$

Also for this solution,

$$\sum_{n=1}^{\infty}(n - \tfrac{1}{2})\, a_n \cos(n - \tfrac{1}{2} x) = -\frac{\partial}{\partial x}\int_c^{x}\frac{g(t)\, dt}{\sqrt{(\cos t - \cos x)}} \tag{7.46}$$

Similarly the equations

$$\sum_{n=1}^{\infty} a_n \cos(n - \tfrac{1}{2})x = f(x), \qquad 0 \leq x < c \tag{7.47}$$

$$\sum_{n=1}^{\infty}(n - \tfrac{1}{2})a_n \cos(n - \tfrac{1}{2})x = 0, \qquad c < x \leq \pi \tag{7.48}$$

have the solution

$$a_1 = \frac{1}{\sqrt{2}}\int_0^c g(t)[1 - \cos t]\,dt, \quad a_n = \frac{1}{\sqrt{2}}\int_0^c g(t)\left\{P_{n-2}(\cos t) - P_n(\cos t)\right\}$$

$$(n > 1) \tag{7.49}$$

where

$$g(t) = -\frac{1}{\pi}\frac{d}{dt}\int_0^{t}\frac{f(x)\, dx}{\sqrt{(\cos x - \cos t)}} \tag{7.50}$$

and where for $0 \leq x < c$

$$\sum_{n=1}^{\infty}(n - \tfrac{1}{2})\, a_n \cos(n - \tfrac{1}{2})x = \frac{\partial}{\partial x}\sin x\int_x^{c}\frac{g(t)\, dt}{\sqrt{(\cos x - \cos t)}} \tag{7.51}$$

Finally we note that the solution of the equations

$$\sum_{n=1}^{\infty} a_n \cos(n - \tfrac{1}{2})x = 0, \qquad 0 \leq x < c \tag{7.52}$$

$$\sum_{n=1}^{\infty}(n-\tfrac{1}{2})\, a_n \cos(n-\tfrac{1}{2})x = f(x), \qquad c < x \leqslant \pi \tag{7.53}$$

can be written in the form

$$a_1 = \frac{1}{\sqrt{2}}\int_c^{\pi}(1+\cos u)g(u)du, \quad a_n = \frac{1}{\sqrt{2}}\int_c^{\pi} g(u)\left\{P_n(\cos u) - P_{n-2}(\cos u)\right\}du, \quad (n \geqslant 2) \tag{7.54}$$

where

$$g(t) = -\frac{\sqrt{2}\gamma_2}{\pi}\frac{d}{dt}\int_t^{\pi}\frac{du}{\sqrt{(\cos t - \cos u)}} - \frac{1}{\pi}\frac{d}{dt}\int_t^{\pi}\frac{du}{\sqrt{(\cos t - \cos u)}}\int_u^{\pi} f(x)dx \tag{7.55}$$

with

$$\gamma_2 = \int_c^{\pi} g(u)\, du,$$

so that

$$\gamma_2\left\{1 - \frac{\sqrt{2}}{\pi}\int_c^{\pi}\frac{du}{\sqrt{(\cos \underline{c} - \cos \underline{u})}}\right\} = \frac{1}{\pi}\int_c^{\pi}\frac{du}{\sqrt{(\cos c - \cos u)}}\int_u^{\pi} f(x)\, dx.$$

For this solution

$$\sum a_n \cos(n-\tfrac{1}{2})x = \sin x\int_c^{x}\frac{g(u)\, du}{\sqrt{(\cos u - \cos x)}}, \quad c < x \leqslant \pi. \tag{7.56}$$

8 Dual Series Relations involving Series of Associated Legendre Functions.

The study of dual series relations in which the series involved are expansions in terms of associated Legendre functions have been studied by Collins (1961), whose treatment we shall follow here.

We begin by considering the problem of the determination of the constants $\{a_n\}$ such that

$$\sum_{n=0}^{\infty} a_n\, T_{m+n}^{-m}(\cos\theta) = f(\theta), \quad 0 \leqslant \theta < \alpha \tag{8.1}$$

$$\sum_{n=0}^{\infty} (2n + 2m + 1)a_n T_{m+n}^{-m}(\cos\theta) = g(\theta), \quad \alpha < \theta \leqslant \pi \tag{8.2}$$

where the functions $f(\theta)$ $g(\theta)$ are prescribed and we assume that $\sin^{-m}\theta\, f(\theta)$ and $\sin^{-m}\theta\, g(\theta)$ are continuous in the closed intervals $[0, \alpha]$, $[\alpha, \pi]$ respectively In §3.4 we saw how dual series equations of this type can arise in the analysis of problems in mathematical physics.

We make the assumption that

$$\sum_{n=0}^{\infty} (2n + 2m + 1)a_n T_{m+n}^{-m}(\cos\theta) = h(\theta), \quad (0 \leqslant \theta \leqslant \alpha) \tag{8.3}$$

then using equations (3.48) and (3.49) we find that

$$a_n = \tfrac{1}{2}(-1)^m \int_0^{\alpha} h(x) T_{m+n}^{m}(\cos x)\sin x\, dx + \tfrac{1}{2}(-1)^m \int_{\alpha}^{\pi} g(x) T_{m+n}^{m}(\cos x)\sin x\, dx, \tag{8.4}$$

$(n = 0, 1, 2, \ldots)$. Substituting this expression for a_n into the series on the left hand side of equation (8.1) and interchanging the order of summation and integration, we find that $h(\theta)$ satisfies the integral equation

$$\tfrac{1}{2}(-1)^m \int_0^{\alpha} h(x) S_m(\theta, x)\sin x\, dx = f(\theta) - \tfrac{1}{2}(-1)^m \int_{\alpha}^{\pi} g(x) S_m(\theta, x)\sin dx, \; 0 < \theta < \alpha \tag{8.5}$$

where the kernel $S_m(\theta, x)$ is defined by the equation

$$S_m(\theta, x) = \sum_{n=0}^{\infty} T_{m+n}^{-m}(\cos\theta)\, T_{m+n}^{m}(\cos x). \tag{8.6}$$

Now, by the addition theorem for Legendre polynomials (MacRobert, 1947, p 328), we have

$$P_r(\cos\Theta) = P_r(\cos\theta)P_r(\cos x) + 2\sum_{m=1}^{r} (-1)^m \cos(m\psi)\, T_r^{-m}(\cos\theta)\, T_r^{m}(\cos x), \tag{8.7}$$

where $\underline{r}$ is a positive integer and

$$\cos\Theta = \cos\theta\cos x + \sin\theta\sin x\cos\psi.$$

From this it follows that

$$\int_0^{2\pi} P_r(\cos\Theta)\cos m\psi\, d\psi = \begin{cases} 2\pi(-1)^m T_r^{-m}(\cos\theta)\, T_r^m(\cos x), & r \geq m \\ 0, & 0 < r < m \end{cases}$$

and hence, from equation (8 6), that

$$S_m(\theta, x) = \frac{(-1)}{2\pi}\sum_{n=0}^{\infty} \int_0^{2\pi} P_{m+n}(\cos\Theta)\cos m\psi\, dy = \frac{(-1)}{2\pi}\sum_{n=0}^{\infty}\int_0^{2\pi} P_n(\cos\Theta)\cos m\psi\, d\psi,$$

Using the fact that

$$\sum_{n=0}^{\infty} P_n(\cos\Theta) = (2 - 2\cos\Theta)^{-\frac{1}{2}}$$

we find on interchanging the order of integration and summation that

$$S_m(\theta, x) = \frac{(-1)^m}{2\pi}\int_0^{2\pi}\frac{\cos(m\psi)\,d\psi}{\sqrt{(2 - 2\cos\Theta)}} = \frac{(-1)^m}{2\pi}\int_0^{2\pi}\frac{\cos(m\psi)\,d\psi}{\sqrt{(s_1^2 + s_2^2 - 2s_1s_2\cos\psi)}},$$

where $s_1 = 2\sin\frac{1}{2}\theta\cos\frac{1}{2}x$, $s_2 = 2\sin\frac{1}{2}x\cos\frac{1}{2}\theta$ and $s_1 > 0$, $s_2 > 0$ for all θ and x since both lie in the open interval $(0, \pi)$. The integral occurring in the equation on the extreme right can be put in another form by using a lemma due to Copson (1947); we find that

$$S_m(\theta, x) = \frac{2(-1)^m}{\pi(s_1s_2)^m}\int_0^{\mathrm{Min}(s_1, s_2)} \frac{s^{2m}\,ds}{\sqrt{\{(s_1^2 - s^2)(s_2^2 - s^2)\}}}. \tag{8.8}$$

If we substitute the form (8 8) for $S_m(\theta, x)$ into equation (8.6) and notice that when $\theta < x$, $s_1 < s_2$ and when $\theta > x$, $s_1 > s_2$, we find that $h(\theta)$ satisfies the equation

$$\frac{\sin^{-m}\theta}{\pi}\left[\int_0^{\theta} h(x)\sin^{1-m}x\,dx\int_0^{s_2}\frac{s^{2m}ds}{\sqrt{[(s_1^2 - s^2)(s_2^2 - s^2)]}} + \int_\theta^{\alpha} h(x)\sin^{1-m}x\,dx \times \right.$$
$$\left.\int_0^{s_1}\frac{s^{2m}\,ds}{\sqrt{[(s_1^2 - s^2)(s_2^2 - s^2)]}}\right]$$

$$= f(\theta) - \frac{\sin^{-m}\theta}{\pi}\int_{\alpha}^{\pi} g(x)\sin^{1-m} x\, dx \int_{0}^{s_1} \frac{s^{2m}\, ds}{\sqrt{[(s_1^2 - s^2)(s_2^2 - s^2)]}}, \; (0 < \theta < \alpha). \quad (8.9)$$

We now change the variable in the inner integral from $\underline{s}$ to $\underline{u}$ where

$$s = 2\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}x\tan\tfrac{1}{2}u$$

and invert the orders of integration in the integrals in equation (8.9) to obtain the integral equation

$$\int_0^{\theta}\frac{H(u)\;(\tan\frac{1}{2}u)^m\,du}{\sqrt{(\cos u - \cos\theta)}} = (\tan\tfrac{1}{2}\theta)^m f(\theta) - \int_0^{\theta}\frac{G(u,\alpha)(\tan\frac{1}{2}u)^{2m}du}{\sqrt{(\cos u - \cos\theta)}} \; (0 < \theta < \alpha) \quad (8.10)$$

where the functions $H(u)$ and $G(u,\alpha)$ are defined by the equations

$$H(u) = \frac{1}{2\pi}\int_u^{\alpha}\frac{h(x)\;(\cot\frac{1}{2}x)^m \sin x\, dx}{\sqrt{(\cos u - \cos x)}}, \quad (8.11)$$

$$G(u,\alpha) = \frac{1}{2\pi}\int_{\alpha}^{\pi}\frac{g(x)\;(\cot\frac{1}{2}x)^m \sin x\, dx}{\sqrt{(\cos u - \cos x)}} \quad (u < \alpha). \quad (8.12)$$

From equations (3.42$\underline{a}$ and $\underline{b}$) we see that the solution of the integral equation (8.10) is

$$H(u) = \frac{(\cot\frac{1}{2}u)^m}{\pi}\frac{d}{du}\int_0^u\frac{\sin\theta(\tan\frac{1}{2}\theta)^m f(\theta)d\theta}{\sqrt{(\cos\theta - \cos u)}} - G(u,\alpha), \; (0 < u < \alpha) \quad (8.13)$$

and from equations (3.43$\underline{a}$ and $\underline{b}$) that the formula giving $h(x)$ in terms of the function $H(u)$ so determined is

$$h(x) = -2(\tan\tfrac{1}{2}x)^m \operatorname{cosec} x\,\frac{d}{dx}\int_x^{\alpha}\frac{H(u)\sin u\,du}{\sqrt{(\cos x - \cos u)}}, (0 < x < \alpha). \quad (8.14)$$

The corresponding value of a_n is obtained by inserting this form for $h(x)$ into equation (8.4). However, it is possible to determine a_n in terms of $H(u)$. Collins has shown that

$$a_n = \frac{(2\pi)^{\frac{1}{2}}\Gamma(n+2m+1)}{\Gamma(n+m+\frac{1}{2})}\left\{\int_0^{\alpha} H(u)\;(\sin\tfrac{1}{2}u)^m\cos(\tfrac{1}{2}u)\,\Pi_{m,n}(\cos u)du \right.$$
$$\left. + \int_0^{\pi} G_0(u)\;(\sin\tfrac{1}{2}u)^m\cos(\tfrac{1}{2}u)\,\Pi_{m,n}(\cos u)du\right\} \quad (8.15)$$

where $\Pi_{m\,n}$ denotes the polynomial defined by the equation

$$\Pi_{m\,n}(\mu) = \frac{\Gamma(n+m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(m+\frac{1}{2})}\,{}_2F_1(-n, n+2m+1;\, m+\tfrac{1}{2};\, \tfrac{1}{2}(1-\mu)) \tag{8.16}$$

and

$$G_0(u) = \begin{cases} G(u, \alpha), & (0 \leqslant u < \alpha), \\ G(u, u), & (\alpha < u \leqslant \pi) \end{cases} \tag{8.17}$$

The cases m = 0, 1 are of most interest from the point of view of applications. Putting m = 0 in these formulae we see that the equations

$$\sum_{n=0}^{\infty} a_n P_n(\cos\theta) = f(\theta), \qquad 0 \leqslant \theta < \alpha \tag{8.18}$$

$$\sum_{n=0}^{\infty} (2n+1) a_n P_n(\cos\theta) = g(\theta), \qquad \alpha < \theta \leqslant \pi \tag{8.19}$$

have solution

$$a_n = \sqrt{2}\left\{\int_0^{\alpha} H(u)\cos\left[(n+\tfrac{1}{2})u\right]du + \int_0^{\pi} G_0(u)\cos\left[(n+\tfrac{1}{2})u\right]du\right\} \tag{8.20}$$

where

$$H(u) = \frac{1}{\pi}\frac{d}{du}\int_0^u \frac{\sin\theta\, f(\theta)\, d\theta}{\sqrt{(\cos\theta - \cos u)}} - G(u, \alpha), \quad G(u, \;) = \frac{1}{2\pi}\int_\alpha^\pi \frac{g(x)\sin x\, dx}{\sqrt{(\cos u - \cos x)}}, \tag{8.21}$$

Also the auxiliary function h(x) is given by the equation

$$h(x) = -2\,\mathrm{cosec}\, x\,\frac{d}{dx}\int_x^\alpha \frac{\sin u\, H(u)\, du}{\sqrt{(\cos x - \cos u)}}, \qquad 0 < x < \alpha. \tag{8.22}$$

As a special case of these results we have that the solution of the equations

$$\sum_{n=0}^{\infty} a_n P_n(\cos\theta) = 1, \qquad 0 \leqslant \theta < \alpha \tag{8.23}$$

$$\sum_{n=0}^{\infty} (2n+1)\, a_n P_n(\cos\theta) = 0, \qquad \alpha < \theta \leqslant \pi \tag{8.24}$$

is

$$a_n = \frac{1}{\pi}\left\{\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right\}. \tag{8.25}$$

The corresponding expression for $H(u)$ is $\frac{\sqrt{2}}{\pi}\cos\frac{1}{2}u$ so that when $0 \leqslant \theta < \alpha$ we have from equation (8.22) that

$$\sum_{n=0}^{\infty}(2n+1)a_n P_n(\cos\theta) = -\frac{2\sqrt{2}}{\pi}\operatorname{cosec}\theta\frac{d}{d\theta}\int_{\theta}^{\alpha}\frac{\sin u\cos(\frac{1}{2}u)\,du}{\sqrt{(\cos\theta - \cos u)}}.$$

The integrations are elementary and we find that

$$\sum_{n=0}^{\infty}(2n+1)a_n P_n(\cos\theta) = \frac{1}{\pi}\left\{\frac{1}{2}\pi - \sin^{-1}\left(\frac{\cos\frac{1}{2}\alpha}{\cos\frac{1}{2}\theta}\right) - \frac{\cos\frac{1}{2}\alpha}{\sqrt{(\cos^2\frac{1}{2}\theta - \cos^2\frac{1}{2}\alpha)}}\right\}, 0 \leqslant \theta < \alpha. \tag{8.26}$$

Similarly if we put $m = 1$ in the general formulae we find that the pair of equations

$$\sum_{n=0}^{\infty} a_n T_{n+1}^{-1}(\cos\theta) = f(\theta), \quad 0 \leqslant \theta < \alpha, \tag{8.27}$$

$$\sum_{n=0}^{\infty}(2n+3)a_n T_{n+1}^{-1}(\cos\theta) = g(\theta), \quad \alpha < \theta \leqslant \pi, \tag{8.28}$$

possesses the solution

$$a_n = \frac{1}{\sqrt{2}}\int_0^{\alpha} H(u)\tan(\tfrac{1}{2}u)\sec(\tfrac{1}{2}u)\left\{(n+1)\sin(n+2)u + (n+2)\sin(n+1)u\right\}du$$

$$+ \int_0^{\pi} G_0(u)\tan(\tfrac{1}{2}u)\sec(\tfrac{1}{2}u)\left\{(n+1)\sin(n+2)u + (n+2)\sin(n+1)u\right\}du \tag{8.29}$$

where

$$H(u) = \frac{2\cot\frac{1}{2}u}{\pi}\frac{d}{du}\int_0^u \frac{\sin^2(\frac{1}{2}\theta)f(\theta)\,d\theta}{\sqrt{(\cos\theta - \cos u)}} - G(u, \alpha) \tag{8.30}$$

$$G(u, \alpha) = \frac{1}{\pi}\int_{\alpha}^{\pi}\frac{g(\theta)\cos^2(\frac{1}{2}\theta)\,d\theta}{\sqrt{(\cos\theta - \cos u)}} \tag{8.31}$$

and $G_0(u)$ is defined by equations (8.17)

We now consider the equations

$$\sum_{n=0}^{\infty} (1 + H_n)\, a_n\, T^{-m}_{m+n}(\cos\theta) = f(\theta), \qquad (0 < \theta < \alpha), \tag{8.32}$$

$$\sum_{n=0}^{\infty} (2n + 2m + 1)\, a_n\, T^{-m}_{m+n}(\cos\theta) = g(\theta), \qquad (\alpha < \theta < \pi), \tag{8.33}$$

where H_n is a given function of n which is $O(n^{-1})$ for large values of n. As before we assume that $\sin^{-m}\theta\, f(\theta)$ is continuous in $(0, \alpha)$ and that $\sin^{-m}\theta\, g(\theta)$ is continuous in (α, π). If we make the assumption (8.3) then the coefficients a_n are again given by equation (8.4); if we insert the value (8.4) for a_n into the series on the left hand side of equation (8.32) we obtain the equation

$$\tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} T^{-m}_{m+n}(\cos\theta)(1 + H_n) \int_0^{\alpha} h(x)\, T^{m}_{m+n}(\cos x)\sin x\, dx$$

$$= f(\theta) - \tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} (1 + H_n)\, T^{-m}_{m+n}(\cos\theta) \int_{\alpha}^{\pi} g(x)\, T^{m}_{m+n}(\cos x)\sin x\, dx, (0 < \theta < \alpha). \tag{8.34}$$

As in the derivation of equation (8.10) we find that

$$\tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} T^{-m}_{m+n}(\cos\theta)\int_{\alpha}^{\pi} g(x) T^{m}_{m+n}(\cos x)\sin x\, dx = \cot^m \tfrac{1}{2}\theta \int_0^{\theta} \frac{G(u, \alpha)\tan^{2m}\frac{1}{2}u\, du}{\sqrt{(\cos u - \cos\theta)}}, \tag{8.35}$$

$$\tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} T^{-m}_{m+n}(\cos\theta)\int_0^{\alpha} h(x) T^{m}_{m+n}(\cos x)\sin x\, dx = \cot^m \tfrac{1}{2}\theta \int_0^{\theta} \frac{H(u)\, \tan^{2m}\frac{1}{2}u\, du}{\sqrt{(\cos u - \cos\theta)}}, \tag{8.36}$$

where $H(u)$ and $G(u, \alpha)$ are defined by equations (8.11) and (8.12) respectively.

If we write

$$I = \tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} H_n\, T^{-m}_{m+n}(\cos\theta) \int_0^{\alpha} h(x)\, T^{m}_{m+n}(\cos x)\sin x\, dx$$

then, using equations (3.47) and (3.50), we find that

$$I = \tfrac{1}{2}\sum_{n=0}^{\infty} \frac{H_n \Gamma(n + 2m + 1)}{\Gamma(n + 1)} \cot^m \tfrac{1}{2}\theta \int_0^{\theta} \frac{R^m_{m+n}(u)\, du}{\sqrt{(\cos u - \cos\theta)}} \int_0^{\alpha} h(x)\sin x \cot^m \tfrac{1}{2}x\, dx \times$$

$$\times \int_0^{x} \frac{R^m_{m+n}(v)\, dv}{\sqrt{(\cos v - \cos x)}},$$

It can be shown that

$$I = \cot^m \tfrac{1}{2}\theta \int_0^\theta \frac{\tan^m \frac{1}{2} u \, du}{\sqrt{(\cos u - \cos\theta)}} \int_0^\alpha \tan^m \tfrac{1}{2} v \, K_1(u, v) H(v) \, dv \tag{8.37}$$

where

$$K_1(u, v) = \pi \cot^m \tfrac{1}{2} u \cot^m \tfrac{1}{2} v \sum_{n=0}^{\infty} \frac{\Gamma(n + 2m + 1)}{\Gamma(n + 1)} H_n R^m_{m+n}(u) R^m_{m+n}(v). \tag{8.38}$$

Similarly if we write

$$J = \tfrac{1}{2}(-1)^m \sum_{n=0}^{\infty} H_n T^{-m}_{m+n}(\cos\theta) \int_\alpha^\pi g(x) \, T^m_{m+n}(\cos x) \sin x \, dx \tag{8.39}$$

it can be shown that if

$$G_0(v) = \begin{cases} G(v, \alpha), & 0 < v < \alpha \\ G(v, v), & \alpha < v < \pi \end{cases} \tag{8.40}$$

then

$$J = \cot^m \tfrac{1}{2}\theta \int_0^\theta \frac{\tan^m \frac{1}{2} u \, du}{\sqrt{(\cos u - \cos\theta)}} \int_0^\pi \tan^m \tfrac{1}{2} v \, K_1(u, v) G_0(v) \, dv. \tag{8.41}$$

If we substitute from equations (8.35), (8.36), (8.37), (8.41) into equation (8.34) we find after some manipulation that the function

$$J(u) = \tan^m \tfrac{1}{2} u \left\{ H(u) + G(u, \alpha) \right\} \tag{8.42}$$

satisfies the Fredholm integral equation of the second kind

$$J(u) + \int_0^\alpha K_1(u, v) \, J(v) \, dv = \frac{\cot^m \frac{1}{2} u}{\pi} \frac{d}{du} \int_0^u \frac{f(x) \tan^m \frac{1}{2} x \sin x \, dx}{\sqrt{(\cos x - \cos u)}}$$

$$- \int_\alpha^\pi \tan^m \tfrac{1}{2} v \, K_1(u, v) \, G(v, v) \, dv, \quad (0 < u < \alpha) \tag{8.43}$$

with symmetric kernel $K_1(u, v)$ defined by equation (8.38). Once we have found the solution of this equation we can determine $H(u)$ from equation (8.42) and the function $h(x)$ from equation (8.14).

The case m = 0 is of special interest. We then have

$$J(u) = H(u) + G(u, \alpha) \tag{8.44}$$

where J(u) is an even function of u satisfying the integral equation

$$J(u) + \frac{1}{\pi}\int_{-\alpha}^{\alpha} K_0(u-v)\, J(v)\, dv = \frac{1}{\pi}\frac{d}{du}\int_0^u \frac{f(x)\sin x\, dx}{\sqrt{(\cos x - \cos u)}}$$

$$-\frac{2}{\pi}\sum_{n=0}^{\infty} H_n \cos(n+\tfrac{1}{2})u\int_{\alpha}^{\pi} G(v, v)\cos(n+\tfrac{1}{2})v\, dv, \quad |u| < \alpha \tag{8.45}$$

where

$$K_0(x) = \sum_{n=0}^{\infty} H_n \cos(n+\tfrac{1}{2})x. \tag{8.46}$$

9. Dual Equations involving Series of Jacobi Polynomials

Recently Noble (1963) and Srivastav (1963c) have considered dual series relations involving Jacobi polynomials. In this section we shall follow the method of solution due to Srivastav.

We consider the equations

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(\alpha+n+1)} A_n P_n^{(\alpha,\beta)}(\cos\theta) = f(\theta), \qquad 0 \leqslant \theta < \phi \tag{9.1}$$

$$\sum_{n=0}^{\infty} \frac{\Gamma(\beta+n+\frac{3}{2})}{\Gamma(\beta+n+1)} A_n P_n^{(\alpha,\beta)}(\cos\theta) = g(\theta), \qquad \phi < \theta \leqslant \pi \tag{9.2}$$

where $\alpha > -\frac{1}{2}$, $\beta > -1$ and $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial defined by the equation

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\, {}_2F_1(-n, n+\alpha+\beta+1;\ \alpha+1;\ \tfrac{1}{2}-\tfrac{1}{2}x) \tag{9.3}$$

or, alternatively, by the equation

$$P_n^{(\alpha,\beta)}(x) = (-1)^n\binom{n+\beta}{n}\, {}_2F_1(-n, n+\alpha+\beta+1;\ \beta+1;\ \tfrac{1}{2}+\tfrac{1}{2}x). \tag{9.4}$$

We shall also require the fact that if $\alpha > -1, \beta > -1$,

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\,dx = p_n^{(\alpha,\beta)} \delta_{mn}, \tag{9.5}$$

where

$$p_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}, \tag{9.6}$$

We observe that if the solution of the pair of dual equations (9.1) and (9.2) is known we can derive the solution of the pair

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+\frac{3}{2})}{\Gamma(\alpha+n+1)} A_n P_n^{(\alpha,\beta)}(\cos\theta) = f(\theta), \quad 0 \leqslant \theta < \phi \tag{9.7}$$

$$\sum_{n=0}^{\infty} \frac{\Gamma(\beta+n+\frac{1}{2})}{\Gamma(\beta+n+1)} A_n P_n^{(\alpha,\beta)}(\cos\theta) = g(\theta), \quad \phi < \theta \leqslant \pi \tag{9.8}$$

by a simple change of variables. For that reason we shall consider only the pair (9.1) and (9.2).

If in equations (8.1) and (8.2) we make the substitutions

$$T_{m+n}^{-m}(\cos\theta) = \frac{\Gamma(n+1)\sin^m\theta}{2^m\Gamma(n+m+1)} P_n^{(m,m)}(\cos\theta)$$

$$a_n = \frac{\Gamma(m+n+\frac{1}{2})}{\Gamma(n+1)} A_n, \quad f(\theta) = 2^{-m}\sin^m\theta\, F(\theta)$$

$$g(\theta) = 2^{1-m}\sin^m\theta\, G(\theta)$$

we find that these equations can be written in the form

$$\sum_{n=0}^{\infty} \frac{\Gamma(m+n+\frac{1}{2})}{\Gamma(m+n+1)} A_n P_n^{(m,m)}(\cos\theta) = F(\theta), \quad 0 \leqslant \theta < \alpha$$

$$\sum_{n=0}^{\infty} \frac{\Gamma(m+n+\frac{3}{2})}{\Gamma(m+n+1)} A_n P_n^{(m,m)}(\cos\theta) = G(\theta), \quad \alpha < \theta \leqslant \pi$$

so that the equations (8.1) and (8.2) are special cases of the equations (9.1) and (9.2) with $\alpha = \beta = m$.

We begin by considering the integral

$$I_1(u) = \int_0^u \frac{\sin x\, P_n^{\alpha,\beta}(\cos x)\,(\sin \tfrac{1}{2} x)^{2\alpha}}{\sqrt{(\cos x - \cos u)}}\,dx, \quad 0 \leqslant u \leqslant \pi . \tag{9.9}$$

If we replace the Jacobi polynomial by the hypergeometric function given by equation (9.3) and interchange the order of summation and integration we find that

$$I_1(u) = 2^{-\alpha}\pi^{\frac{1}{2}}\,\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+n+\frac{3}{2})}\,(1-\cos u)^{\alpha+\frac{1}{2}}\,P_n^{\alpha+\frac{1}{2},\,\beta-\frac{1}{2}}(\cos u). \tag{9.10}$$

Similarly we can show that the integral

$$I_2(v) = \int_v^\pi \frac{\sin x\, P_n^{\alpha,\beta}(\cos x)\,(\cos \tfrac{1}{2} x)^{2\beta}\,dx}{\sqrt{(\cos v - \cos x)}}, \tag{9.11}$$

has the value

$$I_2(v) = 2^{-\beta}\pi^{\frac{1}{2}}\,(1+\cos v)^{\beta+\frac{1}{2}}\,\frac{\Gamma(\beta+n+1)}{\Gamma(\beta+n+\frac{3}{2})}\,P_n^{\alpha-\frac{1}{2},\,\beta+\frac{1}{2}}(\cos v). \tag{9.12}$$

From equations (9.10) and (9.12) it follows immediately as a result of simple differentiations that

$$\frac{d}{du} I_1(u) = 2^{-\alpha}\pi^{\frac{1}{2}}\,\frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+\frac{1}{2}+n)}\,(1-\cos u)^{\alpha-\frac{1}{2}}\sin u\, P_n^{\alpha-\frac{1}{2},\,\beta+\frac{1}{2}}(\cos u), \tag{9.13}$$

and that

$$-\frac{d}{dv} I_2(v) = 2^{-\beta}\pi^{\frac{1}{2}}\,\frac{\Gamma(\beta+1+n)}{\Gamma(\beta+\frac{1}{2}+n)}\,(1+\cos v)^{\beta-\frac{1}{2}}\sin v\, P_n^{\alpha+\frac{1}{2},\,\beta-\frac{1}{2}}(\cos v). \tag{9.14}$$

Using the orthogonality condition (9.5) we can easily show that if

$$f(\cos v) = \sum_{n=0}^{\infty} c_n P_n^{(\alpha,\beta)}(\cos v), \quad 0 \leqslant v \leqslant \pi, \tag{9.15}$$

then the coefficients c_n are given by the equation

$$c_n = \frac{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}\int_0^\pi (\sin\tfrac{1}{2}v)^{2\alpha+1}(\cos\tfrac{1}{2}v)^{2\beta+1} f(\cos v)\times$$
$$\times P_n^{(\alpha,\beta)}(\cos v)\,dv. \qquad (9.16)$$

We are now in a position to solve the pair of equations (9.1), (9.2). If we multiply both sides of (9.1) by $\sin\theta(\sin\frac{1}{2}\theta)^{2\alpha}(\cos\theta - \cos u)^{-\frac{1}{2}}$ and integrate with respect to θ from 0 to u, then differentiate with respect to u, we find on using equation (9.13) that

$$2^{-\alpha}\pi^{\frac{1}{2}}(1-\cos u)^{\alpha-\frac{1}{2}}\sin u\sum_{n=0}^{\infty} A_n P_n^{\alpha-\frac{1}{2},\beta+\frac{1}{2}}(\cos u) = F(u), \quad 0 \leqslant u \leqslant \phi \qquad (9.17)$$

where F(u) is defined in terms of $f(\theta)$ by the equation

$$F(u) = \frac{d}{du}\int_0^u \sin\theta(\sin\tfrac{1}{2}\theta)^{2\alpha}(\cos\theta - \cos u)^{-\frac{1}{2}} f(\theta)\,d\theta. \qquad (9.18)$$

Similarly if we multiply both sides of equation (9.2) by $\sin\theta(\cos\frac{1}{2}\theta)^{2\beta}$ $(\cos u - \cos\theta)^{-\frac{1}{2}}$ and integrate from u to π we find that

$$2^{-\beta}\pi^{\frac{1}{2}}(1+\cos u)^{\beta+\frac{1}{2}}\sum_{n=0}^{\infty} A_n P_n^{(\alpha-\frac{1}{2},\beta+\frac{1}{2})}(\cos u) = G(u), \qquad (9.19)$$

where G(u) is defined in terms of $g(\theta)$ by the equation

$$G(u) = \int_u^\pi \sin\theta(\cos\tfrac{1}{2}\theta)^{2\beta}(\cos u - \cos\theta)^{-\frac{1}{2}} g(\theta)\,d\theta, \quad \phi < u \leqslant \pi \qquad (9.20)$$

If we now make use of the formula (9.16) we find that

$$A_n = \frac{(2\pi)^{-\frac{1}{2}} n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+n+\frac{1}{2})\Gamma(\beta+n+\frac{3}{2})}\times$$
$$\times\Big\{(\cos\tfrac{1}{2}\phi)^{2\beta+1} P_n^{\alpha-\frac{1}{2},\beta+\frac{1}{2}}(\cos\phi)\int_0^\phi \sin\theta(\sin\tfrac{1}{2}\theta)^{2\alpha}(\cos\theta-\cos\phi)^{-\frac{1}{2}} f(\theta)\,d\theta$$
$$+(n+\beta+\tfrac{1}{2})\int_0^\phi \sin\theta(\sin\tfrac{1}{2}\theta)^{2\alpha} f(\theta)\,d\theta\int_\theta^\phi (\cos\theta-\cos v)^{-\frac{1}{2}}(\cos\tfrac{1}{2}v)^{2\beta}\sin\tfrac{1}{2}v\times$$
$$P_n^{\alpha+\frac{1}{2},\beta-\frac{1}{2}}(\cos v)\,dv$$

$$+\int_{\phi}^{\pi}\sin\theta(\cos\tfrac{1}{2}\theta)^{2\beta}g(\theta)d\theta\int_{\phi}^{\theta}(\cos v-\cos\theta)^{-\frac{1}{2}}(\cos\tfrac{1}{2}v)(\sin\tfrac{1}{2}v)^{2\alpha}$$

$$P_n^{\alpha-\frac{1}{2},\,\beta+\frac{1}{2}}(\cos v)\,dv\Big\} \qquad (9.21)$$

If the forms of the functions f and g are complicated the determination of the coefficients A_n by means of equation (9.21) is obviously a very difficult procedure, but in many physical problems it is sufficient to know the behaviour of the functions

$$f_1(\theta)=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(\alpha+n+1)}A_n P_n^{(\alpha,\beta)}(\cos\theta), \qquad \phi<\theta\leqslant\pi \qquad (9.22)$$

$$g_1(\theta)=\sum_{n=0}^{\infty}\frac{\Gamma(\beta+n+\frac{3}{2})}{\Gamma(\beta+n+1)}A_n P_n^{(\alpha,\beta)}(\cos\theta), \qquad 0\leqslant\theta<\phi \qquad (9.23)$$

We can write equation (9.17) in the form

$$\sum_{n=0}^{\infty}A_n P_n^{(\alpha-\frac{1}{2},\,\beta+\frac{1}{2})}(\cos u)=2^{\alpha}\pi^{-\frac{1}{2}}(1-\cos u)^{\frac{1}{2}-\alpha}\operatorname{cosec}u\,F(u),$$

If we multiply both sides of this equation by $(\sin u)(\sin\frac{1}{2}u)^{2\alpha-1}\times$ $(\cos u-\cos x)^{-\frac{1}{2}}$, $\phi<x\leqslant\pi$ and integrate we obtain the relation

$$2^{\frac{1}{2}-\alpha}\pi^{\frac{1}{2}}(1-\cos\theta)^{\alpha}f_1(\theta)=2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\int_0\frac{F(u)\,du}{\sqrt{(\cos u-\cos\theta)}}$$

$$+(2\pi)^{-\frac{1}{2}}\int_{\phi}^{\theta}\sin u(\sin\tfrac{1}{2}u)^{2\alpha-1}(\cos\tfrac{1}{2}u)^{-2\beta-1}\frac{G(u)\,du}{\sqrt{(\cos\theta-\cos u)}}$$

i.e. when $\phi<\theta\leqslant\pi$

$$f_1(\theta)=2^{\alpha}\pi(1-\cos\theta)^{-\alpha}\int_0^{\theta}\frac{F(u)\,du}{\sqrt{(\cos u-\cos\theta)}}$$

$$+2^{\alpha-1}\pi^{-1}(1-\cos\theta)^{-\alpha}\int_{\phi}^{\theta}\sin u(\sin\tfrac{1}{2}u)^{2\alpha-1}(\cos\tfrac{1}{2}u)^{-2\beta-1}\frac{G(u)\,du}{\sqrt{(\cos\theta-\cos u)}}, \qquad (9.24)$$

where F(u) and G(u) are defined by equations (9.18) and (9.20) respectively.

Similarly for $0\leqslant\theta<\phi$ we have

$$f_2(\theta)=-2^{\beta}\pi^{-1}(1+\cos\theta)^{-\beta}\operatorname{cosec}\theta\frac{d}{d\theta}\int_{\theta}^{\phi}\frac{(\cos\frac{1}{2}u)^{2\beta}\sin u(\sin\frac{1}{2}u)^{-2\alpha}}{\sqrt{(\cos\theta-\cos u)}}F(u)du$$

$$-2^{\beta}\pi^{-1}(1+\cos\theta)^{-\beta}\operatorname{cosec}\theta\frac{d}{d\theta}\int_{\phi}^{\pi}\frac{\sin u\, G(u)\, du}{\sqrt{(\cos\theta-\cos u)}}\,. \tag{9.25}$$

REFERENCES

W. D. Collins, 1961, "On some dual series equations and their application to electrostatic problems for spheroidal caps," Proc. Cambridge Phil. Soc., 57, 367.

J. C. Cooke and C. J. Tranter, 1959, "Dual Fourier-Bessel Series," Quart. Journ. Mech. and Appl. Math., (Oxford), 12, 379.

E. T. Copson, 1947, "On the Problem of the Electrified Disc," Proc. Edinb. Math. Soc., (2), 8, 14.

L. Fox and E. T. Goodwin, 1953, "The Numerical Solution of Non-Singular Linear Integral Equations," Phil. Trans. A., 245, 501.

H. Lamb, 1953, "Hydrodynamics," 6th edit., University Press, Cambridge.

T. M. MacRobert, 1947, "Spherical Harmonics," 2nd ed., Methuen and Co., London.

B. Noble, 1963, "Some Dual Series Equations involving Jacobi Polynomials," Proc. Cambridge Phil. Soc. (to be published).

E. Reissner, 1937, "Freie und erzwungene Torsions-schwingungen des elastischen Halbraumes," Ing. Archiv., 8, 229.

I. N. Sneddon, 1951, "Fourier Transforms," McGraw-Hill Book Co., New York.

1956, "The Special Functions of Mathematical Physics and Chemistry, Interscience, New York.

1962, "Fractional Integration and Dual Integral Equations," (North Carolina State College, Applied Mathematics Research Group, Report).

I. N. Sneddon and R. P. Srivastav, 1963, "Dual Series Relations: I: Dual Relations Involving Fourier-Sessel Series." Proc. Roy. Soc. Edinburgh (to be published).

R. P. Srivistav, 1963a, "Dual Series Relations II: Dual Relations involving Dini Series" Proc. Roy. Soc. Edinburgh (to be published).

1963b, "Dual Series Relations III: Dual Relations involving Trigonometrical Series," Proc. Roy. Soc. Edinburgh (to be published).

1963c, "Dual Series Relations IV: Dual Relations involving Series of Jacobi polynomials." Proc. Roy. Soc. Edinburgh (to be published).

1963d, "A Note on the Solution of an Integral Equation," Proc. Edinburgh Math. Soc. (to be published).

C. J. Tranter, 1959a, "On the Analogies between some Series containing Bessel Functions and Certain Special Cases of the Weber-Schaftheittin Integral," Quart. Journ. Math. (Oxford) 10, 110.

1959b. "Dual Trigonometrical Series," Proc. Glasgow Math. Assocn..

G. N. Watson, 1944, "A Treatise on the Theory of Bessel Functions," University Press, Cambridge.

E. T. Whittaker and G. N. Watson, 1927, "Modern Analysis," 4th edit., University Press, Cambridge.

W. E. Williams, 1963, "The Solution of Dual Series and Dual Integral Equations," Proc. Glasgow Math. Soc., (to be published).